



University
of Glasgow

Strachan, I.A.B. (2010) *Weyl groups and elliptic solutions of the WDVV equations*. *Advances in Mathematics*, 224 (5). pp. 1801-1838. ISSN 0001-8708

<http://eprints.gla.ac.uk/61428/>

Deposited on: 28 June 2012

WEYL GROUPS AND ELLIPTIC SOLUTIONS OF THE WDVV EQUATIONS

IAN A. B. STRACHAN

ABSTRACT. A functional ansatz is developed which gives certain elliptic solutions of the Witten-Dijkgraaf-Verlinde-Verlinde (or WDVV) equation. This is based on the elliptic trilogarithm function introduced by Beilinson and Levin. For this to be a solution results in a number of purely algebraic conditions on the set of vectors that appear in the ansatz, this providing an elliptic version of the idea, introduced by Veselov, of a \mathcal{V} -system.

Rational and trigonometric limits are studied together with examples of elliptic \mathcal{V} -systems based on various Weyl groups. Jacobi group orbit spaces are studied: these carry the structure of a Frobenius manifold. The corresponding ‘almost dual’ structure is shown, in the A_N and B_N and conjecturally for an arbitrary Weyl group, to correspond to the elliptic solutions of the WDVV equations.

Transformation properties, under the Jacobi group, of the elliptic trilogarithm are derived together with various functional identities which generalize the classical Frobenius-Stickelberger relations.

CONTENTS

1. Introduction	2
1.1. Frobenius Manifolds and almost-duality	2
1.2. Examples	4
2. The Elliptic Polylogarithm and its Properties	7
2.1. Notation	7
2.2. The elliptic polylogarithm	8
3. Transformation properties of the WDVV equations	11
3.1. Analysis of the WDVV equations	12
3.2. Modular transformations of the structure functions	12
3.3. Periodicity properties of the structure functions	14
4. Singularity properties	15
5. The Main Theorem	18
5.1. Rational and Trigonometric Limits	19
6. Examples of elliptic \mathcal{V} -systems	20
6.1. The case $\mathcal{U} = \mathcal{R}_W$	21
6.2. The case $\mathcal{U} = \mathcal{R}_W \cup \mathcal{R}_W^{irreg}$	22
7. Frobenius-Stickelberger Identities	26
8. Jacobi Group Orbit Spaces	27
8.1. Jacobi groups and Jacobi forms	28

Date: November 4, 2009.

1991 *Mathematics Subject Classification.* 11F55, 53B50, 53D45.

Key words and phrases. Frobenius manifolds, WDVV equations, Jacobi groups, elliptic functions, elliptic polylogarithms.

8.2. Hurwitz spaces	30
9. Comments	33
Acknowledgments	34
References	34

1. INTRODUCTION

One recurrent theme in the theory of integrable systems is the tower of generalizations

$$\text{rational} \longrightarrow \text{trigonometric} \longrightarrow \text{elliptic},$$

the paradigm being provided by the Calogero-Moser system, where the original rational interaction term may be generalized

$$\frac{1}{z^2} \longrightarrow \frac{1}{\sin^2 z} \longrightarrow \wp(z)$$

whilst retaining integrability. A second recurrent theme is the appearance of root systems, the paradigm being again provided by the Calogero-Moser system where the interaction term

$$\sum_{i \neq j} \frac{1}{(z_i - z_j)^2}$$

can, on fixing the centre of mass, be written as

$$\sum_{\alpha \in \mathcal{R}_{A_N}} \frac{1}{(\alpha, \mathbf{z})^2},$$

where the sum is taken over the roots \mathcal{R}_{A_N} of the A_N Coxeter group [24]. The integrability of the system is preserved if other root systems are used.

These two themes occur in many other integrable structures; R matrices, quantum groups, Dunkl operators, KZ-equations all admit (to a greater or lesser extent) rational, trigonometric and elliptic versions and generalizations to arbitrary root systems (see for example [8] and the references therein). In this paper elliptic solutions of the Witten-Dijkgraaf-Verlinde-Verlinde (or WDVV) equations will be studied for arbitrary Weyl groups, these sitting at the right of the following tower of generalizations:

$$\begin{array}{ccccc} \mathbb{C}^N/W & \longrightarrow & \mathbb{C}^{N+1}/\widetilde{W} & \longrightarrow & \Omega/J(\mathfrak{g}). \\ \left\{ \begin{array}{c} \text{Coxeter group} \\ \text{orbit space} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{Extended affine Weyl} \\ \text{orbit space} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{Jacobi group} \\ \text{orbit space} \end{array} \right\} \end{array}$$

We begin by defining a Frobenius manifold.

1.1. Frobenius Manifolds and almost-duality.

Definition 1. *An algebra $(\mathcal{A}, \circ, \eta, e)$ over \mathbb{C} is a Frobenius algebra if:*

- *the algebra $\{\mathcal{A}, \circ\}$ is commutative, associative with unity e ;*
- *the multiplication is compatible with a \mathbb{C} -valued bilinear, symmetric, non-degenerate inner product*

$$\eta : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}$$

in the sense that

$$\eta(a \circ b, c) = \eta(a, b \circ c)$$

for all $a, b, c \in \mathcal{A}$.

With this structure one may define a Frobenius manifold [9]:

Definition 2. (M, \circ, e, η, E) is a Frobenius manifold if each tangent space $T_p M$ is a Frobenius algebra varying smoothly over M with the additional properties:

- the inner product is a flat metric on M (the term ‘metric’ will denote a complex-valued quadratic form on M).
- $\nabla e = 0$, where ∇ is the Levi-Civita connection of the metric;
- the tensor $(\nabla_W \circ)(X, Y, Z)$ is totally symmetric for all vectors $W, X, Y, Z \in TM$;
- the vector field E (the Euler vector field) has the properties

$$\nabla(\nabla E) = 0$$

and the corresponding one-parameter group of diffeomorphisms acts by conformal transformations of the metric and by rescalings on the Frobenius algebras $T_p M$.

Since the metric η is flat there exists a distinguished coordinate system (defined up to linear transformations) of so-called flat coordinates¹ $\{t^\alpha, \alpha = 0, \dots, N+1\}$ in which the components of the metric are constant. From the various symmetry properties of tensors \circ and $\nabla \circ$ it then follows that there exists a function F , the prepotential, such that in the flat coordinate system,

$$\begin{aligned} c_{\alpha\beta\gamma} &= \eta \left(\frac{\partial}{\partial t^\alpha} \circ \frac{\partial}{\partial t^\beta}, \frac{\partial}{\partial t^\gamma} \right), \\ &= \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}, \end{aligned}$$

and the associativity condition then implies that the pair (F, η) satisfy the WDVV-equations

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\gamma \partial t^\delta} - \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\gamma \partial t^\alpha} = 0,$$

where $\alpha, \beta, \gamma, \delta = 0 \dots, N+1$.

Consider the vector field E^{-1} defined by the condition

$$E^{-1} \circ E = e.$$

This is defined on $M^* = M \setminus \Sigma$, where Σ is the discriminant submanifold on which E^{-1} is undefined. With this field one may define a new ‘dual’ multiplication $\star : TM^* \times TM^* \rightarrow TM^*$ by

$$X \star Y = E^{-1} \circ X \circ Y, \quad \forall X, Y \in TM^*.$$

This new multiplication is clearly commutative and associative, with the Euler vector field being the unity field for the new multiplication.

Furthermore, this new multiplication is compatible with the intersection form g on the Frobenius manifold, i.e.

$$g(X \star Y, Z) = g(X, Y \star Z), \quad \forall X, Y, Z \in TM^*.$$

¹This labeling is for future notational convenience.

Here g is defined by the equation

$$g(X, Y) = \eta(X \circ Y, E^{-1}), \quad \forall X, Y \in TM^*$$

(and hence is well-defined on M^*). Alternatively one may use the metric η to extend the original multiplication to the cotangent bundle and define

$$g^{-1}(x, y) = \iota_E(x \circ y), \quad \forall x, y \in T^*M^*.$$

The intersection form has the important property that it is flat, and hence there exists a distinguished coordinate system $\{\mathbf{p}\}$ in which the components of the intersection form are constant. It turns out that there exists a dual prepotential F^* such that its third derivatives give the structure functions c_{ijk}^* for the dual multiplication. More precisely [10]:

Theorem 3. *Given a Frobenius manifold M , there exists a function F^* defined on M^* such that:*

$$\begin{aligned} c_{ijk}^* &= g\left(\frac{\partial}{\partial p^i} \star \frac{\partial}{\partial p^j}, \frac{\partial}{\partial p^k}\right), \\ &= \frac{\partial^3 F^*}{\partial p^i \partial p^j \partial p^k}. \end{aligned}$$

Moreover, the pair (F^*, g) satisfy the WDVV-equations in the flat coordinates $\{\mathbf{p}\}$ of the metric g .

Thus given a specific Frobenius manifold one may construct a ‘dual’ solution to the WDVV-equations by constructing the flat-coordinates of the intersection form and using the above result to find the tensor c_{ijk}^* from which the dual prepotential may be constructed.

1.2. Examples. The simplest class of Frobenius manifolds is given by the so-called Saito construction on the space of orbits of a Coxeter group. Let W be an irreducible Coxeter group acting on a real vector space V of dimension N . The action extends to the complexified space $V \otimes \mathbb{C}$. The orbit space

$$V \otimes \mathbb{C}/W \cong \mathbb{C}^N/W$$

has a particularly nice structure, this following from Chevalley’s theorem on the ring of W -invariant polynomials:

Theorem 4. *There exists a set of W -invariant polynomial $s_i(\mathbf{z}), i = 1, \dots, N$ such that*

$$\mathbb{C}[z_1, \dots, z_N]^W \cong \mathbb{C}[s_1, \dots, s_N].$$

On this orbit space one may define a metric (a complex-valued quadratic form) by taking the Lie-derivative of the W -invariant Euclidean metric g on $V \otimes \mathbb{C}$

$$\eta^{-1} = \mathcal{L}_e g^{-1}$$

where e is a vector field constructed from the highest degree invariant polynomial. It was proved by K. Saito that this metric is non-degenerate and flat [28]. One therefore obtains a flat pencil of metrics from which one may construct a polynomial solution - polynomial in the flat coordinates of the metric η - to the WDVV equations.

The dual prepotential for this class of Frobenius manifolds is particularly simple:

$$(1) \quad F = \frac{1}{4} \sum_{\alpha \in \mathcal{R}_W} (\alpha, \mathbf{z})^2 \log(\alpha, \mathbf{z})^2$$

where the sum is taken over the roots of the Coxeter group W [20, 21, 22]. However, the space of solutions of the same functional form is far larger. Veselov [32] derived the algebraic conditions, known as \vee -conditions, on the set of vectors \mathfrak{U} that are required for the prepotential

$$F = \frac{1}{4} \sum_{\alpha \in \mathfrak{U}} (\alpha, \mathbf{z})^2 \log(\alpha, \mathbf{z})^2$$

to satisfy the WDVV equations (we assume throughout this paper that if $\alpha \in \mathfrak{U}$ then $-\alpha \in \mathfrak{U}$ automatically). What is required here is a refinement of this idea, namely that of a complex Euclidean \vee -system [15]

Definition 5. *Let \mathfrak{h} be a complex vector space with non-degenerate bilinear form (\cdot, \cdot) and let \mathfrak{U} be a collection of vectors in \mathfrak{h} . A complex Euclidean \vee -system \mathfrak{U} satisfies the following conditions:*

- \mathfrak{U} is well distributed, i.e. $\sum_{\alpha \in \mathfrak{U}} h_\alpha(\alpha, \mathbf{u})(\alpha, \mathbf{v}) = 2h_\mathfrak{U}^\vee(\mathbf{u}, \mathbf{v})$ for some λ ;
- on any 2-dimensional plane Π the set $\Pi \cap \mathfrak{U}$ is either well distributed or reducible (i.e. the union of two non-empty orthogonal subsystems).

Note the following:

- the constants h_α could be absorbed into the α . In applications these constants will be both positive and negative. Hence the requirement of a complex vector space.
- the constant $h_\mathfrak{U}^\vee$ can be zero in certain spaces.

One further comment has to be made in the case when $h_\mathfrak{U}^\vee = 0$. We require here that the inverse metric used in the WDVV equations is the non-degenerate bilinear form (\cdot, \cdot) on \mathfrak{h} rather than one - possibly degenerate - constructed from the sum of derivatives of F as used in [14].

Trigonometric solutions were studied in [11], corresponding to extended affine Weyl groups. As in the Coxeter case one has a Chevalley-type theorem and a well defined orbit space on which one may define, following the Saito-construction, a flat metric and hence a solution to the WDVV equations. It is to be expected, though a full proof for arbitrary Weyl groups is currently lacking, that the corresponding dual solutions will take the following functional form

$$(2) \quad F = \text{cubic terms} + \sum_{\alpha \in \mathcal{R}_W} h_\alpha Li_3 \left(e^{i(\alpha, \mathbf{x})} \right)$$

where $Li_3(x)$ is the trilogarithm and h_α are Weyl-invariant sets of constants. Solutions of the WDVV equations of this type have been studied by a number of authors [21, 23] but are only known to be almost dual solutions to the extended affine Weyl Frobenius manifolds in certain special cases (e.g. $W = A_N^{(k)}$) [26]. Trigonometric \vee -conditions, conditions on the vectors α that ensure that the prepotential

$$F = \text{cubic terms} + \sum_{\alpha \in \mathfrak{U}} h_\alpha Li_3 \left(e^{i(\alpha, \mathbf{x})} \right)$$

satisfies the WDVV equations, have also been studied recently [13].

Elliptic solutions were studied in [2], being defined on the Jacobi group orbit space $\Omega/J(\mathfrak{g})$. Further details and definitions will be given in Section 8, following [2],[12] and [33]. The Jacobi group $J(\mathfrak{g})$ (where \mathfrak{g} is a complex finite dimensional simple Lie algebra of rank N with Weyl group W) acts on the space

$$\Omega = \mathbb{C} \oplus \mathfrak{h} \oplus \mathbb{H}$$

where \mathfrak{h} is the complex Cartan subalgebra of \mathfrak{g} and \mathbb{H} is the upper-half-plane, and this leads to the study of invariant functions - the Jacobi forms. Analogous to the Coxeter case, the orbit space

$$\Omega/J(\mathfrak{g})$$

is a manifold and carries the structure of a Frobenius manifold. In [27], using the Hurwitz space description (see Section 8.2)

$$\Omega/J(A_N) \cong H_{1,N+1}(N+1),$$

the dual prepotential was constructed.

Theorem 6. [27] *The intersection form on the space $\Omega/J(A_N)$ is given by the formula*

$$g = 2du d\tau - \sum_{i=0}^N (dz^i)^2 \Big|_{\sum_{j=0}^N z^j=0}$$

(where $u \in \mathbb{C}$, $\mathbf{z} \in \mathfrak{h}$ and $\tau \in \mathbb{H}$). The dual prepotential is given by the formula²

$$\begin{aligned} F^*(u, \mathbf{z}, \tau) &= \frac{1}{2}\tau u^2 - \frac{1}{2}u \sum_{i=0}^N (z^i)^2 \\ &+ \frac{1}{2} \sum'_{i \neq j} \frac{1}{(2\pi i)^3} \left\{ \mathcal{L}i_3(e^{2i(z^i - z^j)}, e^{2\pi i \tau}) - \mathcal{L}i_3(1, e^{2\pi i \tau}) \right\} \\ &- (N+1) \sum_j \frac{1}{(2\pi i)^3} \left\{ \mathcal{L}i_3(e^{2iz^j}, e^{2\pi i \tau}) - \mathcal{L}i_3(1, e^{2\pi i \tau}) \right\}. \end{aligned}$$

where this function is evaluated on the plane $\sum_{j=0}^N z^j = 0$.

The precise definitions of the various terms in these formulae will be given below, but for now we note that this dual prepotential is given in terms of the elliptic trilogarithm $\mathcal{L}i_3(z, q)$ introduced by Beilinson and Levin [1, 19]. This function has appeared already in the theory of Frobenius manifolds in the enumeration of curves [18].

This result is curious - as well as the A_N root vectors appearing in the solution certain extra vectors (in fact weight vectors) appear: these do not appear in the corresponding rational and trigonometric solutions. This work raised a number of questions:

- Is there a direct verification that the function that appears in Theorem 6 satisfies the WDVV equations? Recall that its construction was via a Hurwitz space construction in terms of certain holomorphic maps between the complex torus and the Riemann sphere.
- What is the origin of the ‘extra’ vectors in the solution?

²Note, \sum_j' includes the term $j = 0$.

- Can one construct solutions for other Weyl groups?

The purpose of this paper is to study solutions of the WDVV equations which take the functional form

$$F(u, \mathbf{z}, \tau) = \frac{1}{2}u^2\tau - \frac{1}{2}u(\mathbf{z}, \mathbf{z}) + \sum_{\alpha \in \mathfrak{U}} h_\alpha f(\mathbf{z}_\alpha, \tau),$$

where

$$f(z, \tau) = \frac{1}{(2\pi i)^3} \{ \mathcal{L}i_3(e^{2\pi iz}, e^{2\pi i\tau}) - \mathcal{L}i_3(1, e^{2\pi i\tau}) \},$$

deriving a set of elliptic \vee -conditions on the ‘roots’ contained in the set \mathfrak{U} . Thus the above questions can all be answered affirmatively. This leaves the following question:

- For which elliptic \vee -systems is the solution the almost-dual solution to the Jacobi group orbit space $\Omega/J(\mathfrak{g})$?

This question has been answered already in the A_N case [27] and in this paper we extend the results to the B_N case. For other Weyl groups it remains an open problem.

2. THE ELLIPTIC POLYLOGARITHM AND ITS PROPERTIES

The functional form of the above prepotential uses the elliptic polylogarithm. In this section this is defined and its transformation properties under shifts and modular transformations are studied. Before this we define various special functions and the notation that will be used throughout the rest of this paper.

2.1. Notation. There are, unfortunately, many different definitions and normalizations for elliptic, number-theoretic and other special functions. Here we list the definitions used in this paper. Let $q = e^{2\pi i\tau}$, where $\tau \in \mathbb{H}$.

- ϑ_1 -function:

$$\vartheta_1(z|\tau) = -i(e^{\pi iz} - e^{-\pi iz}) q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{2\pi iz})(1 - q^n e^{-2\pi iz}).$$

The fundamental lattice is generated by $z \mapsto z + 1, z \mapsto z + \tau$, and the function itself satisfies the complex heat equation

$$\frac{\partial^2 \vartheta_1}{\partial z^2} = 4\pi i \frac{\partial \vartheta_1}{\partial \tau}.$$

- Bernoulli numbers and Bernoulli polynomials:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad B_n(z) = \sum_{k=0}^n \binom{n}{k} B_k z^{n-k}.$$

- Eisenstein series:

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad k \in 2\mathbb{N}$$

where $\sigma_k(n) = \sum_{d|n} d^k$.

- Dedekind η -function:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

- Polylogarithm function:

$$Li_N(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^N}, \quad |z| < 1.$$

Note that ϑ_1 , E_2 and η are related:

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{2\pi i}{24} E_2(\tau) = \frac{1}{12\pi i} \frac{\vartheta_1'''(0, \tau)}{\vartheta_1'(0, \tau)}.$$

These have the following properties under inversion of the independent variable:

$$\begin{aligned} \tau^{-n} E_n\left(-\frac{1}{\tau}\right) &= E_n(\tau), \quad n \geq 4; \\ \tau^{-2} E_2\left(-\frac{1}{\tau}\right) &= E_2(\tau) + \frac{12}{2\pi i \tau}; \\ \eta\left(-\frac{1}{\tau}\right) &= \sqrt{\frac{\tau}{i}} \eta(\tau) \end{aligned}$$

where in the last formula the square-root is taken to have non-negative real part. The polylogarithm has the inversion property (for $n \in \mathbb{N}$ - other more complicated versions hold for other values):

$$(3) \quad Li_n(e^{2\pi i z}) + (-1)^n Li_n(e^{-2\pi i z}) = -\frac{(2\pi i)^n}{n!} B_n(z).$$

This inversion formula holds: if $\Im(z) \geq 0$ for $0 \leq \Re(z) < 1$, and if $\Im(z) < 0$ for $0 < \Re(z) \leq 1$. This, and other similar formulae, may be used to analytically continue the function outside the unit disc to a multi-valued holomorphic function on $\mathbb{C} \setminus \{0, 1\}$. For a discussion of the monodromy of the polylogarithm function see [25]. This multivaluedness will also occur in the solution of the WDVV equations. However this multivaluedness occurs in the quadratic terms only and hence any physical quantities are single-valued.

2.2. The elliptic polylogarithm. An ‘obvious’ elliptic generalization of the polylogarithm function is

$$\mathcal{L}i_r(\zeta, q) = \sum_{n=-\infty}^{\infty} Li_r(q^n \zeta).$$

However this series diverges, but by using the inversion formula (3) and ζ -function regularization one can arrive at the following definition of the elliptic polylogarithm function [1, 19]:

$$\mathcal{L}i_r(\zeta, q) = \sum_{n=0}^{\infty} Li_r(q^n \zeta) + \sum_{n=1}^{\infty} Li_r(q^n \zeta^{-1}) - \chi_r(\zeta, q), \quad r \text{ odd},$$

where

$$\chi_r(\zeta, q) = \sum_{j=0}^r \frac{B_{j+1}}{(r-j)!(j+1)!} (\log \zeta)^{(r-j)} (\log q)^j.$$

A real-valued version of this function had previously been studied by Zagier [35]. With this the function f may be defined.

Definition 7. *The function $f(z, \tau)$, where $z \in \mathbb{C}, \tau \in \mathbb{H}$, is defined to be:*

$$f(z, \tau) = \frac{1}{(2\pi i)^3} \{ \mathcal{L}i_3(e^{2\pi i z}, q) - \mathcal{L}i_3(1, q) \}.$$

It follows from the definitions that

$$(4) \quad \left(\frac{d}{d\tau} \right)^3 \frac{1}{(2\pi i)^3} \mathcal{L}i_3(1, q) = \frac{1}{120} E_4(\tau)$$

and

$$(5) \quad \left(\frac{\partial}{\partial z} \right)^2 f(z, \tau) = -\frac{1}{2\pi i} \log \left\{ \frac{\vartheta_1(z, \tau)}{\eta(\tau)} \right\}.$$

Thus the elliptic-trilogarithm may be thought of as a classical function (or, at least, a neoclassical function) as it may be obtained from classical functions via nested integration and other standard procedures. It does, however, provide a systematic way to deal with the arbitrary functions that would appear this way. The notation $F \simeq G$ will be used if the functions F and G differ by a quadratic function in the variables $\{u, \mathbf{z}, \tau\}$ (recall that any prepotential satisfying the WDVV equations is only defined up to quadratic terms in the flat-coordinates).

Proposition 8. *The function f has the following transformation properties:*

$$\begin{aligned} f(z+1, \tau) &\simeq f(z, \tau); \\ f(z, \tau+1) &\simeq f(z, \tau); \\ f(z+\tau, \tau) &\simeq f(z, \tau) + \left\{ \frac{1}{6}z^3 + \frac{1}{4}z^2\tau + \frac{1}{6}z\tau^2 + \frac{1}{24}\tau^3 \right\}; \\ f(-z, \tau) &\simeq f(z, \tau). \end{aligned}$$

The function also has the alternative expansions:

$$(6) \quad \begin{aligned} f(z, \tau) &\simeq -\frac{1}{(2\pi i)^3} \left\{ \frac{1}{2}z^2 \log z + z^2 \log \eta(\tau) \right\} \\ &+ \frac{1}{(2\pi i)^3} \sum_{n=1}^{\infty} \frac{(-1)^n E_{2n}(\tau) B_{2n}}{(2n+2)!(2n)} (2\pi z)^{2n+2} \end{aligned}$$

and

$$(7) \quad \begin{aligned} f(z, \tau) &\simeq \frac{1}{(2\pi i)^3} \mathcal{L}i_3(e^{2\pi i z}) + \frac{1}{12}z^3 - \frac{1}{24}z^2\tau \\ &- \frac{4}{(2\pi i)^3} \sum_{r=1}^{\infty} \left\{ \frac{q^r}{(1-q^r)} \right\} \frac{\sin^2(\pi r z)}{r^3} \end{aligned}$$

Furthermore,

$$f\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) \simeq \frac{1}{\tau^2} f(z, \tau) - \frac{1}{\tau^3} \frac{z^4}{4!}.$$

Proof The first two relations follow from the definition. The third and fourth use the inversion formula for polylogarithms (3).

The proof of (6) and (7) just involves some careful resummation. Consider the first two terms in the definition of f :

$$\begin{aligned} \sum_{n=0}^{\infty} Li_3(q^n e^{2\pi iz}) + \sum_{n=1}^{\infty} Li_3(q^n e^{-2\pi iz}) &= Li_3(e^{2\pi iz}) + \\ &2 \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s)!} \left\{ \sum_{n,r=1}^{\infty} q^{nr} r^{2s-3} \right\} (2\pi z)^{2s}. \end{aligned}$$

From this series (7) follows immediately. To obtain (6) one rearranges the terms. The $s = 0$ term cancels in the final expression and the remaining terms may be re-expressed in terms of Eisenstein series (for $s > 1$) or the Dedekind function (for $s = 1$). Finally, using the result

$$\begin{aligned} \frac{1}{(2\pi i)^3} \frac{d^3}{dz^3} Li_3(e^{2\pi iz}) &= -\frac{1}{2} [1 + \coth(\pi iz)] , \\ &= -\left[\frac{1}{2} + \frac{1}{(2\pi z)} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (2\pi iz)^{2n-1} \right] \end{aligned}$$

one may obtain a series for $Li_3(e^{2\pi iz})$. Putting all these parts together gives the series (6). \square

Theorem 9. *The function*

$$h(z, \tau) = f(2z, \tau) - 4f(z, \tau)$$

satisfies the partial differential equation

$$h^{(3,0)}(z, \tau) h^{(1,2)}(z, \tau) - \left[h^{(2,1)}(z, \tau) \right]^2 + 4h^{(0,3)}(z, \tau) = 0$$

where

$$h^{(m,n)}(z, \tau) = \frac{\partial^{n+m} h}{\partial z^m \partial \tau^n}.$$

Proof Let

$$\Delta(z, \tau) = h^{(3,0)}(z, \tau) h^{(1,2)}(z, \tau) - \left[h^{(2,1)}(z, \tau) \right]^2 + 4h^{(0,3)}(z, \tau).$$

Using the transformation properties in Proposition 8 one may derive the transformation properties of the derivatives and hence for the combination Δ . While the individual terms have quite complicated transformation properties, those for Δ are very simple:

$$\Delta(z+1, \tau) = \Delta(z, \tau),$$

$$\Delta(z+\tau, \tau) = \Delta(z, \tau),$$

$$\Delta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = \tau^4 \Delta(z, \tau).$$

The first two of these equations imply that the function Δ is doubly periodic. From the series expansion in Proposition 8 it follows that Δ has no poles: the only term which has a pole is $f^{(3,0)}$ and this cancels with the zero in $f^{(1,2)}$. Thus Δ is doubly

periodic with no poles and hence must be a function of τ -alone (i.e. a theta-constant). The remaining transformation property implies that Δ is a modular function of degree 4. The q -series representation of the function f in Proposition 8 implies that Δ is actually a cusp-form. But the space of cusp-forms of degree 4 is empty and hence $\Delta = 0$. \square

Corollary 10. *The pair*

$$\begin{aligned} F(u, z, \tau) &= \frac{1}{2}u^2\tau - uz^2 + h(z, \tau), \\ &= \frac{1}{2}u^2\tau - uz^2 + f(2z, \tau) - 4f(z, \tau) \end{aligned}$$

and

$$g = 2du d\tau - 2dz^2$$

satisfy the WDVV equations.

Proof The WDVV equations for the above prepotential reduce to the single equation $\Delta = 0$ so the result follows immediately from the above Theorem. \square

Using the same methods it is straightforward to show that the function

$$(8) \quad \tilde{h}(z, \tau) = \frac{3}{4(2\pi i)^3} \left[\mathcal{L}i_3(e^{2\pi iz}, q) + \frac{3}{2}\mathcal{L}i_3(1, q) \right]$$

also satisfies the equation $\Delta = 0$. It is, however, the function h which defines the dual prepotential to the A_1 -Jacobi group orbit space - see Theorem 6.

3. TRANSFORMATION PROPERTIES OF THE WDVV EQUATIONS

Recall that we seek a solution of the WDVV of the form

$$(9) \quad F(u, \mathbf{z}, \tau) = \frac{1}{2}u^2\tau - \frac{1}{2}u(\mathbf{z}, \mathbf{z}) + \sum_{\alpha \in \mathfrak{U}} h_\alpha f(\mathbf{z}_\alpha, \tau)$$

with $f(z, \tau)$ being given by Definition 7. Sometimes the notation \mathbf{z}_α will be used to denote (\mathbf{z}, α) , especially for terms involving the function f . Thus $f((\mathbf{z}, \alpha), \tau)$ will be written $f(\mathbf{z}_\alpha, \tau)$ or even $f(\mathbf{z}_\alpha)$. The coordinates $\{t^\alpha, \alpha = 0, 1, \dots, N, N+1\}$ are defined to be

$$\begin{aligned} t^0 &= u, \\ t^i &= z^i, \quad i = 1, \dots, N, \\ t^{N+1} &= \tau. \end{aligned}$$

Latin indices will range from 1 to N and Greek from 0 to $N+1$ so the dimension of the manifold is $N+2$, with $N \geq 1$. In addition $u \in \mathbb{C}, \mathbf{z} \in \mathfrak{h} \cong \mathbb{C}^N, \tau \in \mathbb{H}$, so $(u, \mathbf{z}, \tau) \in \Omega$. Later, \mathfrak{h} will be the complex Cartan subalgebra of a simple Lie algebra \mathfrak{g} of rank N with Weyl group W , but for now it may be thought of a just \mathbb{C}^N . Also, $(,)$ denotes the standard Euclidean inner product on \mathfrak{h} . It follows from the functional dependence on $t^0 = u$ that ∂_u is the unity vector field and hence the metric on Ω is

$$g = d\tau du + du d\tau - (d\mathbf{z}, d\mathbf{z}).$$

One of the main ideas of this paper is to extend Theorem 9 to higher dimension, using doubly-periodicity and modular arguments to prove that the WDVV equations are satisfied. To begin we require a detailed analysis of the WDVV equations themselves.

3.1. Analysis of the WDVV equations. The WDVV equations are the conditions for a commutative algebra to be associative. Thus they may be written in terms of the vanishing of the associator

$$\Delta[X, Y, Z] = (X \circ Y) \circ Z - X \circ (Y \circ Z).$$

Since in the case being considered we have a unity element these simplify further: if any of the vector field is equal to the unity field then Δ vanishes identically. Since the vector field $\partial_\tau \in T\mathbb{H}$ is special (for example, it behaves differently to the other variables under modularity transformation), we decompose these equation further, taking the inner product with arbitrary vector fields to obtain scalar-valued equations.

Proposition 11. *The WDVV equations for a multiplication with unity field are equivalent to the vanishing of the following functions:*

$$\begin{aligned} \Delta^{(1)}(\mathbf{u}, \mathbf{v}) &= g(\partial_\tau \circ \partial_\tau, \mathbf{u} \circ \mathbf{v}) - g(\partial_\tau \circ \mathbf{u}, \partial_\tau \circ \mathbf{v}), \\ \Delta^{(2)}(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= g(\partial_\tau \circ \mathbf{u}, \mathbf{v} \circ \mathbf{w}) - g(\partial_\tau \circ \mathbf{w}, \mathbf{u} \circ \mathbf{v}), \\ \Delta^{(3)}(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}) &= g(\mathbf{u} \circ \mathbf{v}, \mathbf{w} \circ \mathbf{x}) - g(\mathbf{u} \circ \mathbf{x}, \mathbf{v} \circ \mathbf{w}) \end{aligned}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \in T\mathfrak{h}$.

In terms of coordinate vector fields these conditions are:

$$\begin{aligned} \Delta_{ij}^{(1)} &= g_{ij} c_{\tau\tau\tau} + g^{pq} \{c_{\tau\tau p} c_{ijq} - c_{\tau ip} c_{\tau jq}\}, \\ \Delta_{ijk}^{(2)} &= \{g_{jk} c_{\tau\tau i} - g_{ij} c_{\tau\tau k}\} + g^{pq} \{c_{\tau ip} c_{jkq} - c_{\tau kp} c_{ijq}\}, \\ \Delta_{ijrs}^{(3)} &= \{g_{ij} c_{\tau rs} + g_{rs} c_{\tau ij} - g_{is} c_{\tau rj} - g_{rj} c_{\tau is}\} + g^{pq} \{c_{ijp} c_{rsq} - c_{isp} c_{rjq}\} \end{aligned}$$

where $g_{ij} = -(\partial_i, \partial_j)$. The function Δ in theorem (9) is, since $\dim_{\mathbb{C}} \mathfrak{h} = 1$, proportional to $\Delta^{(1)}(\mathbf{x}, \mathbf{x})$.

3.2. Modular transformations of the structure functions.

Lemma 12. *Let*

$$(10) \quad \begin{aligned} \hat{u} &= u - \frac{(\mathbf{z}, \mathbf{z})}{2\tau}, \\ \hat{\mathbf{z}} &= \frac{\mathbf{z}}{\tau}, \\ \hat{\tau} &= -\frac{1}{\tau}. \end{aligned}$$

Then

$$F(\hat{u}, \hat{\mathbf{z}}, \hat{\tau}) = \frac{1}{\tau^2} \left\{ F(u, \mathbf{z}, \tau) - \frac{1}{2} u (2u\tau - (\mathbf{z}, \mathbf{z})) \right\}$$

if and only if

$$(11) \quad \sum_{\alpha \in \mathfrak{U}} h_\alpha(\alpha, \mathbf{z})^4 = 3(\mathbf{z}, \mathbf{z})^2.$$

Proof This follows immediately from Proposition 8. \square

The origin of the transformation (10) comes from the study of symmetries of the WDVV equations (see [9] Appendix B). A symmetry is a transformation

$$\begin{aligned} t^\alpha &\mapsto \hat{t}^\alpha, \\ g_{\alpha\beta} &\mapsto \hat{g}_{\alpha\beta}, \\ F &\mapsto \hat{F} \end{aligned}$$

that acts on the solution space of the WDVV equations. In particular, (10) is just the transformation, denoted I in [9],

$$\begin{aligned} \hat{t}^0 &= \frac{1}{2} \frac{t_\sigma t^\sigma}{t^{N+1}}, \\ \hat{t}^i &= \frac{t^i}{t^{N+1}}, \quad i = 1, \dots, N, \\ \hat{t}^{N+1} &= -\frac{1}{t^{N+1}}, \\ \hat{g}_{\alpha\beta} &= g_{\alpha\beta}, \\ \hat{F}(\hat{\mathbf{t}}) &= (t^{N+1})^{-2} \left[F(\mathbf{t}) - \frac{1}{2} t^0 (t_\sigma t^\sigma) \right] \end{aligned}$$

which induces a symmetry of the WDVV equations. Up to a simple equivalence, $I^2 = \mathbb{I}$. It follows from this and Lemma 12 that we are at the fixed point of this involution and that, rather than telling one how to construct a new solution from a seed solution, it gives the transformation property of the various functions under the modular transformations. A simple modification of Lemma B.1 [9] immediately gives:

Proposition 13. *Suppose F is given by equation (9) where condition (11) is assumed to hold. Let $c_{\alpha\beta\gamma} = \partial_\alpha \partial_\beta \partial_\gamma F(\mathbf{t})$. Then*

$$c_{\alpha\beta\gamma}(\mathbf{z}, \tau + 1) = c_{\alpha\beta\gamma}(\mathbf{z}, \tau)$$

and

$$\begin{aligned} c_{ijk} \left(\frac{\mathbf{z}}{\tau}, -\frac{1}{\tau} \right) &= \tau c_{ijk}(\mathbf{z}, \tau) - g_{ij} z_k - g_{jk} z_i - g_{ki} z_j, \\ c_{\tau ij} \left(\frac{\mathbf{z}}{\tau}, -\frac{1}{\tau} \right) &= \tau c_{ij\alpha}(\mathbf{z}, \tau) t^\alpha - \frac{1}{2} g_{ij} (t_\sigma t^\sigma) - z_i z_j, \\ c_{\tau \tau i} \left(\frac{\mathbf{z}}{\tau}, -\frac{1}{\tau} \right) &= \tau c_{i\alpha\beta}(\mathbf{z}, \tau) t^\alpha t^\beta - z_i (t_\sigma t^\sigma), \\ c_{\tau \tau \tau} \left(\frac{\mathbf{z}}{\tau}, -\frac{1}{\tau} \right) &= \tau c_{\alpha\beta\gamma}(\mathbf{z}, \tau) t^\alpha t^\beta t^\gamma - \frac{3}{4} (t_\sigma t^\sigma)^2. \end{aligned}$$

With these, the transformation properties of the functions $\Delta^{(i)}$ are

$$\Delta^{(i)}(\mathbf{z}, \tau + 1) = \Delta^{(i)}(\mathbf{z}, \tau), \quad i = 1, 2, 3$$

and

$$\begin{aligned}
\Delta_{ijrs}^{(3)}\left(\frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right) &= \tau^2 \Delta_{ijrs}^{(3)}(\mathbf{z}, \tau), \\
\Delta_{ijk}^{(2)}\left(\frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right) &= \tau^3 \Delta_{ijk}^{(2)}(\mathbf{z}, \tau) + \tau^2 z^r \Delta_{irkj}^{(3)}(\mathbf{z}, \tau), \\
\Delta_{ij}^{(1)}\left(\frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right) &= \tau^4 \Delta_{ij}^{(1)}(\mathbf{z}, \tau) - \tau^3 z^r \left\{ \Delta_{ijr}^{(2)}(\mathbf{z}, \tau) + \Delta_{jir}^{(2)}(\mathbf{z}, \tau) \right\} \\
&\quad + \tau^2 z^a z^b \Delta_{abij}^{(3)}(\mathbf{z}, \tau).
\end{aligned}$$

Proof The proof is straightforward and uses the transformation properties of f derived in Proposition 8. \square

It is important to note that these $\Delta^{(i)}$ are powers series, not Laurent series, in the q -variable. Again, this follows from the q -expansion of f given in proposition 8.

3.3. Periodicity properties of the structure functions. We assume that there exists a vector $\mathbf{p} \in \mathfrak{h}$ such that $(\alpha, \mathbf{p}) \in \mathbb{Z}$ for all $\alpha \in \mathfrak{U}$. Later we will require the existence of a full N -dimensional lattice (the ‘weight lattice’ associated to the ‘roots’ in \mathfrak{U}), but for now we just require a single such vector. From Proposition 8 it follows that

$$f((\alpha, \mathbf{z} + \mathbf{p}), \tau) \simeq f((\alpha, \mathbf{z}), \tau)$$

and hence $F(u, \mathbf{z} + \mathbf{p}, \tau) \simeq F(u, \mathbf{z}, \tau)$. Thus

$$\Delta^{(i)}(\mathbf{z} + \mathbf{p}, \tau) = \Delta^{(i)}(\mathbf{z}, \tau), \quad i = 1, 2, 3.$$

The calculation of the transformations under shifts $\mathbf{z} \mapsto \mathbf{z} + \mathbf{p}\tau$ requires more care.

Proposition 14. *Assume that the following conditions hold:*

$$\sum_{\alpha \in \mathfrak{U}} h_{\alpha}(\alpha, \mathbf{z})^4 = 3(\mathbf{z}, \mathbf{z})^2,$$

and $(\alpha, \mathbf{p}) \in \mathbb{Z}$ for all $\alpha \in \mathfrak{U}$. Then

$$\begin{aligned}
h(\mathbf{z} + \mathbf{p}\tau, \tau) &\simeq h(\mathbf{z}, \tau) + \\
&\quad \frac{1}{8} \left\{ \begin{aligned} &4(\mathbf{p}, \mathbf{z})(\mathbf{z}, \mathbf{z}) + \tau [4(\mathbf{p}, \mathbf{z})^2 + 2(\mathbf{p}, \mathbf{p})(\mathbf{z}, \mathbf{z})] \\ &+ 4\tau^2(\mathbf{p}, \mathbf{z})(\mathbf{p}, \mathbf{p}) + \tau^3(\mathbf{p}, \mathbf{p})^2 \end{aligned} \right\}
\end{aligned}$$

where

$$h(\mathbf{z}, \tau) = \sum_{\alpha \in \mathfrak{U}} h_{\alpha} f(\mathbf{z}_{\alpha}, \tau).$$

Proof From Proposition 8 it follows by induction, for $n \in \mathbb{Z}$, that

$$f(z + n\tau, \tau) \simeq f(z, \tau) + \frac{1}{24} (4nz^3 + 6n^2\tau z^2 + 4n^3\tau^2 z + n^4\tau^3)$$

and since, by assumption, $(\alpha, \mathbf{p}) \in \mathbb{Z}$, it follows immediately that

$$\begin{aligned}
f((\alpha, \mathbf{z}) + (\alpha, \mathbf{p})\tau, \tau) &\simeq f((\alpha, \mathbf{z}), \tau) + \\
&\quad \frac{1}{24} \left\{ \begin{aligned} &4(\alpha, \mathbf{p})(\alpha, \mathbf{z})^3 + 6\tau(\alpha, \mathbf{p})^2(\alpha, \mathbf{z})^2 + \\ &4\tau^2(\alpha, \mathbf{p})^3(\alpha, \mathbf{z}) + \tau^3(\alpha, \mathbf{p})^4 \end{aligned} \right\}.
\end{aligned}$$

Hence, on summing over α ,

$$h(\mathbf{z} + \mathbf{p}\tau, \tau) \simeq h(\mathbf{z}, \tau) + \frac{1}{24} \left\{ \begin{aligned} &4 \sum_{\alpha} h_{\alpha}(\alpha, \mathbf{p})(\alpha, \mathbf{z})^3 + 6\tau \sum_{\alpha} h_{\alpha}(\alpha, \mathbf{p})^2(\alpha, \mathbf{z})^2 + \\ &4\tau^2 \sum_{\alpha} h_{\alpha}(\alpha, \mathbf{p})^3(\alpha, \mathbf{z}) + \tau^3 \sum_{\alpha} h_{\alpha}(\alpha, \mathbf{p})^4 \end{aligned} \right\}.$$

Hence, using the first condition (and its polarized version), the result follows. \square

With this the following Proposition may be proved: the first part is immediate and the second part follows from routine but tedious calculations.

Proposition 15. *Under the conditions of the above proposition, the structure functions have the following transformation properties:*

$$\begin{aligned} c_{ijk}(\mathbf{z} + \mathbf{p}\tau, \tau) &= c_{ijk}(\mathbf{z}, \tau) + p_i g_{jk} + p_j g_{ki} + p_k g_{ij}, \\ c_{\tau ij}(\mathbf{z} + \mathbf{p}\tau, \tau) &= c_{\tau ij}(\mathbf{z}, \tau) - p^a c_{ija}(\mathbf{z}, \tau) - \left(p_i p_j + \frac{1}{2}(\mathbf{p}, \mathbf{p}) g_{ij} \right), \\ c_{\tau \tau i}(\mathbf{z} + \mathbf{p}\tau, \tau) &= c_{\tau \tau i}(\mathbf{z}, \tau) - 2p^a c_{\tau ai}(\mathbf{z}, \tau) + p^a p^b c_{abi}(\mathbf{z}, \tau) + (\mathbf{p}, \mathbf{p}) p_i, \\ c_{\tau \tau \tau}(\mathbf{z} + \mathbf{p}\tau, \tau) &= c_{\tau \tau \tau}(\mathbf{z}, \tau) - 3p^a c_{\tau \tau a}(\mathbf{z}, \tau) + 3p^a p^b c_{\tau ab}(\mathbf{z}, \tau) - p^a p^b p^c c_{abc}(\mathbf{z}, \tau) \\ &\quad - \frac{3}{4}(\mathbf{p}, \mathbf{p})^2. \end{aligned}$$

The $\Delta^{(i)}$ have the following transformation properties:

$$\begin{aligned} \Delta_{ijrs}^{(3)}(\mathbf{z} + \mathbf{p}\tau, \tau) &= \Delta_{ijrs}^{(3)}(\mathbf{z}, \tau), \\ \Delta_{ijk}^{(2)}(\mathbf{z} + \mathbf{p}\tau, \tau) &= \Delta_{ijk}^{(2)}(\mathbf{z}, \tau) + p^a \Delta_{ijka}^{(3)}(\mathbf{z}, \tau), \\ \Delta_{ij}^{(1)}(\mathbf{z} + \mathbf{p}\tau, \tau) &= \Delta_{ij}^{(1)}(\mathbf{z}, \tau) + p^a \left\{ \Delta_{ija}^{(2)}(\mathbf{z}, \tau) + \Delta_{jia}^{(2)}(\mathbf{z}, \tau) \right\} + p^a p^b \Delta_{ijab}^{(3)}(\mathbf{z}, \tau), \end{aligned}$$

We are now in the position to rehearse the main theorem. If we have a full N -dimensional weight lattice, then $\Delta^{(3)}$ is doubly periodic in all \mathbf{z} variables and if we can show it has no poles then it must be a function of τ alone. The modularity properties of $\Delta^{(3)}$ then imply that it must be zero. Repeating the argument sequentially for $\Delta^{(2)}$ and then $\Delta^{(1)}$ will give the desired result. To proceed further requires the examination of the singularity properties of the $\Delta^{(i)}$.

4. SINGULARITY PROPERTIES

To study the singularity properties of the $\Delta^{(i)}$ we require a more detailed analysis of these functions. Using equation (9) and Proposition 11 one obtains:

$$\begin{aligned}
\Delta^{(1)} &= \Delta^{(1)}(\mathbf{u}, \mathbf{v}) \\
&= -(\mathbf{u}, \mathbf{v}) \sum_{\alpha \in \mathfrak{U}} h_\alpha f^{(0,3)}(\mathbf{z}_\alpha, \tau) \\
&\quad + \sum_{\alpha, \beta \in \mathfrak{U}} h_\alpha h_\beta (\alpha, \beta)(\alpha, \mathbf{v}) \left[\begin{array}{l} +(\beta, \mathbf{u}) f^{(2,1)}(\mathbf{z}_\beta, \tau) f^{(2,1)}(\mathbf{z}_\alpha, \tau) \\ -(\alpha, \mathbf{u}) f^{(1,2)}(\mathbf{z}_\beta, \tau) f^{(3,0)}(\mathbf{z}_\alpha, \tau) \end{array} \right]
\end{aligned}$$

$$\begin{aligned}
\Delta^{(2)} &= \Delta^{(2)}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \\
&= \sum_{\alpha \in \mathfrak{U}} h_\alpha [(\mathbf{u}, \mathbf{v})(\alpha, \mathbf{w}) - (\mathbf{w}, \mathbf{v})(\alpha, \mathbf{u})] f^{(1,2)}(\mathbf{z}_\alpha, \tau) \\
&\quad + \sum_{\alpha, \beta \in \mathfrak{U}} h_\alpha h_\beta (\alpha, \beta)(\alpha, \mathbf{v}) [(\alpha \wedge \beta)(\mathbf{u}, \mathbf{w})] f^{(2,1)}(\mathbf{z}_\beta, \tau) f^{(3,0)}(\mathbf{z}_\alpha, \tau)
\end{aligned}$$

$$\begin{aligned}
\Delta^{(3)} &= \Delta^{(3)}(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}) \\
&= \sum_{\alpha \in \mathfrak{U}} h_\alpha \left[\begin{array}{l} +(\alpha, \mathbf{v})(\alpha, \mathbf{w})(\mathbf{u}, \mathbf{x}) - (\alpha, \mathbf{x})(\alpha, \mathbf{w})(\mathbf{u}, \mathbf{v}) \\ +(\alpha, \mathbf{u})(\alpha, \mathbf{x})(\mathbf{v}, \mathbf{w}) - (\alpha, \mathbf{u})(\alpha, \mathbf{v})(\mathbf{w}, \mathbf{x}) \end{array} \right] f^{(2,1)}(\mathbf{z}_\alpha, \tau) \\
&\quad - \frac{1}{2} \sum_{\alpha, \beta \in \mathfrak{U}} h_\alpha h_\beta (\alpha, \beta) [(\alpha \wedge \beta)(\mathbf{u}, \mathbf{w})] [(\alpha \wedge \beta)(\mathbf{v}, \mathbf{x})] f^{(3,0)}(\mathbf{z}_\alpha, \tau) f^{(3,0)}(\mathbf{z}_\beta, \tau),
\end{aligned}$$

where $(\alpha \wedge \beta)(\mathbf{u}, \mathbf{v}) = (\alpha, \mathbf{u})(\beta, \mathbf{v}) - (\alpha, \mathbf{v})(\beta, \mathbf{u})$.

The only derivative of f that gives rise to a pole is the $f^{(3,0)}$ derivative; all other derivatives are analytic in z - this following from (6). Therefore the only parts of the $\Delta^{(i)}$ that could contribute to a singularity are those which contain this derivative.

Proposition 16. *Let Π_α denote a plane through the origin containing the vector α and α^\perp a vector in Π_α perpendicular to α . Then, at $(\alpha, \mathbf{z}) = 0$:*

- $\Delta^{(1)}(\mathbf{u}, \mathbf{v})$ has no pole if the scalar equation

$$(12) \quad \sum_{\beta \in \Pi_\alpha \cap \mathfrak{U}} h_\beta (\alpha, \beta) (\beta, \alpha^\perp)^{2n+1} = 0, \quad n = 1, 2, \dots,$$

holds.

- $\Delta^{(2)}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ has no pole if the bilinear form equation

$$(13) \quad \sum_{\beta \in \Pi_\alpha \cap \mathfrak{U}} h_\beta (\alpha, \beta) (\alpha \wedge \beta)(\beta, \alpha^\perp)^{2n} = 0, \quad n = 1, 2, \dots,$$

holds.

- $\Delta^{(3)}(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x})$ has no pole if the 4-linear form equation

$$(14) \quad \sum_{\beta \in \Pi_\alpha \cap \mathfrak{U}} h_\beta (\alpha, \beta) (\alpha \wedge \beta) \otimes (\alpha \wedge \beta)(\beta, \alpha^\perp)^{2n+1} = 0, \quad n = 1, 2, \dots,$$

holds.

Here

$$(\alpha \wedge \beta)(\mathbf{u}, \mathbf{v}) = (\alpha, \mathbf{u})(\beta, \mathbf{v}) - (\alpha, \mathbf{v})(\beta, \mathbf{u})$$

and

$$(\alpha \wedge \beta) \otimes (\alpha \wedge \beta)(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}) = [(\alpha \wedge \beta)(\mathbf{u}, \mathbf{v})] [(\alpha \wedge \beta)(\mathbf{w}, \mathbf{z})] .$$

Proof The only third derivative of f which contains a pole is the $f^{(3,0)}$ -derivative and hence the only part of $\Delta^{(1)}$ that could contain poles is the term

$$\sum_{\alpha, \beta \in \mathfrak{U}} h_\alpha h_\beta (\alpha, \beta) (\alpha, \mathbf{v})(\alpha, \mathbf{u}) f^{(1,2)}(\mathbf{z}_\beta) f^{(3,0)}(\mathbf{z}_\alpha) .$$

Since $f^{(3,0)}$ only has a simple pole the term involving the poles is, up to a non-zero constant,

$$\sum_{\alpha \in \mathfrak{U}} \frac{h_\alpha (\alpha, \mathbf{u})(\alpha, \mathbf{v})}{(\mathbf{z}, \alpha)} \sum_{\beta \in \mathfrak{U}} h_\beta (\alpha, \beta) f^{(1,2)}(\mathbf{z}_\beta) .$$

The function $f^{(1,2)}(\mathbf{z}_\beta)$ is odd and hence may be written as $\sum_{n=0}^{\infty} A_n(\tau)(\mathbf{z}, \beta)^{2n+1}$ (the explicit expressions for the non-zero functions A_n are not required - they may be derived from Proposition 8). Thus a sufficient condition for the absence of poles is, for arbitrary $\alpha \in \mathfrak{U}$:

$$(\alpha, \mathbf{z}) \text{ divides } \sum_{\alpha \text{ fixed}, \beta \in \mathfrak{U}} h_\beta (\alpha, \beta) (\beta, \mathbf{z})^{2n+1}, \quad n = 0, 1, \dots$$

Note that this is automatically satisfied if $n = 0$ by the first condition in Definition 5. This sum may be rewritten as sums over vectors in 2-planes Π_α containing α , and hence a sufficient condition for the absence of poles is, for arbitrary $\alpha \in \mathfrak{U}$:

$$(\alpha, \mathbf{z}) \text{ divides } \sum_{\beta \in \Pi_\alpha \cap \mathfrak{U}} h_\beta (\alpha, \beta) (\beta, \mathbf{z})^{2n+1}, \quad n = 1, 2, \dots$$

On decomposing each β in the plane Π_α as $\beta = \mu\alpha + \nu\alpha^\perp$ (so $\nu = (\beta, \alpha^\perp)/(\alpha^\perp, \alpha^\perp)$) one finds that all terms in the binomial expansion of $(\beta, \mathbf{z})^{2n+1}$ contain a (α, \mathbf{z}) -term except the final $[\nu(\mathbf{z}, \alpha^\perp)]^{2n+1}$ -term. Thus a sufficient condition for the absence of poles in $\Delta^{(1)}$ is

$$\sum_{\beta \in \Pi_\alpha \cap \mathfrak{U}} h_\beta (\alpha, \beta) (\beta, \alpha^\perp)^{2n+1} = 0, \quad n = 1, 2, \dots$$

The proof of the $\Delta^{(2)}$ condition is identical: $f^{(2,1)}$ is an even function, and the lowest term vanishes on using the first condition in Definition 5.

The function $\Delta^{(3)}$ contains a term

$$\sum_{\alpha, \beta \in \mathfrak{U}} h_\alpha h_\beta (\alpha, \beta) [(\alpha \wedge \beta)(\mathbf{u}, \mathbf{w})] [(\alpha \wedge \beta)(\mathbf{v}, \mathbf{x})] \frac{1}{(\alpha, \mathbf{z})} \frac{1}{(\beta, \mathbf{z})} .$$

This vanishes by definition of a complex Euclidean \vee -system [15]. The proof of the remaining $\Delta^{(3)}$ condition is identical to the above: $f^{(3,0)}$ is an odd function, and the lowest term vanishes on using a polarized version of condition (11). \square

5. THE MAIN THEOREM

We can now draw the various components together, but first we define an elliptic \vee -system.

Definition 17. *Let \mathfrak{U} be a complex Euclidean \vee -system. An elliptic \vee -system is a complex Euclidean \vee -system with the following additional conditions:*

- $\sum_{\alpha \in \mathfrak{U}} h_{\alpha}(\alpha, \mathbf{z})^4 = 3(\mathbf{z}, \mathbf{z})^2$;
- The three conditions in Proposition 16 hold;
- There exists a full N -dimensional weight lattice of vectors \mathbf{p} such that $(\mathbf{p}, \alpha) \in \mathbb{Z}$ for all $\alpha \in \mathfrak{U}$.

Examples of elliptic \vee -systems will be constructed in the next section. With this definition in place one arrives at the main theorem.

Theorem 18. *Let \mathfrak{U} be an elliptic \vee -system. If $h_{\mathfrak{U}}^{\vee} = 0$ then the function*

$$(15) \quad F(u, \mathbf{z}, \tau) = \frac{1}{2}u^2\tau - \frac{1}{2}u(\mathbf{z}, \mathbf{z}) + \sum_{\alpha \in \mathfrak{U}} h_{\alpha} f(\mathbf{z}_{\alpha}, \tau)$$

satisfies the WDVV equations. If $h_{\mathfrak{U}}^{\vee} \neq 0$ then the modified prepotential

$$(16) \quad F \longrightarrow F + \frac{10(h_{\mathfrak{U}}^{\vee})^2}{3(2\pi i)^3} \mathcal{L}i_3(1, q)$$

satisfies the WDVV equations.

Proof From the conditions in the definition of an elliptic \vee -system and Proposition 15 it follows that $\Delta^{(3)}$ is doubly periodic in all \mathbf{z} -variables, and from the conditions in Proposition 16 it follows that it has no poles. It therefore must be a function of τ alone. From Proposition 13 it follows that $\Delta^{(3)}$ is a modular function of degree 2 and from Proposition 13 it follows that it contains only positive powers in its q -expansion. Hence it is a modular form of degree 2 and hence must be zero.

This argument can now be repeated for $\Delta^{(2)}$ (a modular function of degree 3 with only positive powers in its q -expansion and hence a modular form of degree 3 and so must be zero).

Finally, the same arguments implies that $\Delta^{(1)}$ is a modular form of degree 4, and hence it must be a multiple of the modular form $E_4(\tau)$. Thus

$$\Delta^{(1)}(\mathbf{u}, \mathbf{v}) = m(\mathbf{u}, \mathbf{v}) E_4(\tau).$$

To find $m(\mathbf{u}, \mathbf{v})$ one just requires the $O(1)$ -terms in the q -expansion of $\Delta^{(1)}$. On using equation (7) one finds that

$$\begin{aligned} m(\mathbf{u}, \mathbf{v}) &= \sum_{\alpha, \beta \in \mathfrak{U}} h_{\alpha} h_{\beta}(\alpha, \beta)(\alpha, \mathbf{v})(\beta, \mathbf{u}) \left(-\frac{1}{12}\right)^2, \\ &= \frac{1}{36} (h_{\mathfrak{U}}^{\vee})^2 (\mathbf{u}, \mathbf{v}). \end{aligned}$$

Hence

$$\Delta^{(1)} = \frac{1}{36} (h_{\mathfrak{U}}^{\vee})^2 E_4(\tau) (\mathbf{u}, \mathbf{v}).$$

Thus if $h_{\mathfrak{U}}^{\vee} = 0$ then $\Delta^{(i)} = 0$ for $i = 1, 2, 3$ and hence (15) satisfies the WDVV equations.

If $h_{\mathfrak{U}}^{\vee} \neq 0$ one has to modify the ansatz for F :

$$F \longrightarrow F + \mu \frac{1}{(2\pi i)^3} \mathcal{L}i_3(1, q).$$

This change only effects $c_{\tau\tau\tau}$ and hence the above argument on the vanishing of $\Delta^{(3)}$ and $\Delta^{(2)}$ is unchanged. With this new ansatz $\Delta^{(1)}$ undergoes a slight change:

$$\Delta^{(1)} \longrightarrow \Delta^{(1)} - \mu \frac{1}{120} E_4(\tau)(\mathbf{u}, \mathbf{v}),$$

on using (4). Thus if

$$\mu = \frac{10}{3} (h_{\mathfrak{U}}^{\vee})^2$$

then the modified $\Delta^{(1)}$ is zero and hence (16) satisfies the WDVV equations. \square

5.1. Rational and Trigonometric Limits. From the leading order behaviour, obtained from Proposition 8,

$$f = -\frac{1}{(4\pi i)} z^2 \log z \quad \text{as } z \rightarrow 0,$$

and

$$f \simeq \frac{1}{(2\pi i)^3} Li_3(e^{2\pi i z}) + \frac{1}{12} z^3 \quad \text{as } q \rightarrow 0$$

one may obtain rational and trigonometric solutions, of lower dimension, of the WDVV equations.

Proposition 19. *Given an elliptic \vee -system \mathfrak{U} the following are solutions of the WDVV equations:*

Rational limit

$$F^{\text{rational}} = \sum_{\alpha \in \mathfrak{U}} h_{\alpha}(\alpha, \mathbf{z})^2 \log(\alpha, \mathbf{z}).$$

The metric in this case is the standard Euclidean inner product on \mathfrak{h} .

Trigonometric limit I

If $h_{\mathfrak{U}}^{\vee} = 0$ then

$$F^{\text{trig}} = \sum_{\alpha \in \mathfrak{U}} h_{\alpha} Li_3\left(e^{2\pi i(\mathbf{z}, \alpha)}\right).$$

The metric in this case is the standard Euclidean inner product on \mathfrak{h} .

Trigonometric limit II

If $h_{\mathfrak{U}}^{\vee} \neq 0$ then

$$F^{\text{trig}} = \frac{1}{6} u^3 - \frac{1}{2} u(\mathbf{z}, \mathbf{z}) + \frac{1}{(2\pi i)^3} \left(\frac{3}{h_{\mathfrak{U}}^{\vee}}\right)^{\frac{1}{2}} \sum_{\alpha \in \mathfrak{U}} h_{\alpha} Li_3\left(e^{2\pi i(\mathbf{z}, \alpha)}\right).$$

In this case one has a covariantly constant unity vector field ∂_u and hence the metric in this case $g = du^2 - (d\mathbf{z}, d\mathbf{z})$.

The proof just involves the examination of the associator $\Delta^{(3)}$ under the above mentioned limits.

6. EXAMPLES OF ELLIPTIC \vee -SYSTEMS

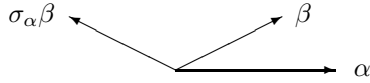
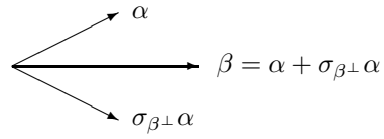
In this section we construct examples of elliptic \vee -systems based on a Weyl group W . Recall that by assumption, if $\alpha \in \mathfrak{U}$ then $-\alpha \in \mathfrak{U}$. We now also assume that the constants h_α are Weyl invariant, i.e. $h_{w(\alpha)} = h_\alpha$ for $w \in W$. We denote the number of vectors in \mathfrak{U} by $|\mathfrak{U}|$. The calculations for specific groups will be done using the standard notion for roots and weights, see for example [16].

Two classes of examples will be given, the first where $\mathfrak{U} = \mathcal{R}_W$ (where \mathcal{R}_W is the root system of W) and the second where $\mathfrak{U} = \mathcal{R}_W \cup \mathcal{R}_W^{irreg}$, where \mathcal{R}_W^{irreg} contains a set of W -invariant vectors that form an irregular orbit under the action of W . We first construct W -invariant sets of vectors (and constants h_α) satisfying the two conditions

$$(17) \quad \sum_{\alpha \in \mathfrak{U}} h_\alpha(\alpha, \mathbf{z})^4 = 3(\mathbf{z}, \mathbf{z})^2,$$

$$(18) \quad \sum_{\alpha \in \mathfrak{U}} h_\alpha(\alpha, \mathbf{u})(\alpha, \mathbf{v}) = 2h_{\mathfrak{U}}^\vee(\mathbf{u}, \mathbf{v})$$

and then check that the conditions in Proposition 16 are satisfied, which will be done with the help of the following lemma. Recall that these conditions involve summing over vectors in the plane $\Pi_\alpha \cap \mathfrak{U}$. In the cases to be discussed here these vectors occur in pairs, related by certain reflections, and the corresponding terms in the sum cancel. Let $\sigma_\alpha \beta$ denote the reflection of the vector β in the line with normal vector α . The pairs in the set of vectors $\Pi_\alpha \cap \mathfrak{U}$ will occur in two types:

Type A: $\alpha \in \mathcal{R}_W$ 

Type B

Type A pairs are very familiar: they occur in Weyl group (indeed, Coxeter group) root systems (with certain special angles). Type B pairs will occur when an extra set of Weyl invariant vectors is appended to the root system - see Section 6.2. Both these types of configuration appear in \vee -systems and deformed root systems [7, 14, 15, 32].

Lemma 20. *Let $\alpha \in \mathfrak{U}$ and suppose that the terms in the sums $\sum_{\beta \in \Pi_\alpha \cap \mathfrak{U}}$ occur in pairs of Type A or Type B. Then the conditions in Proposition 16 are satisfied:*

- (a) *for type A configurations if and only if $h_\beta = h_{\sigma_\alpha \beta}$;*
- (b) *for type B configurations if and only if $(\alpha, \beta)h_\beta = (\alpha, \alpha - \beta)h_{\sigma_{\beta^\perp} \alpha}$.*

Proof

Consider the first condition in Proposition 16, namely equation (12), and consider the partial sums:

Type A

$$\Xi_A = h_\beta(\alpha, \beta)(\alpha^\perp, \beta)^n + h_{\sigma_\alpha \beta}(\alpha, \sigma_\alpha \beta)(\alpha^\perp, \sigma_\alpha \beta)^n;$$

Type B

$$\Xi_B = h_\beta(\alpha, \beta)(\alpha^\perp, \beta)^n + h_{\sigma_{\beta^\perp} \alpha}(\alpha, \sigma_{\beta^\perp} \alpha)(\alpha^\perp, \sigma_{\beta^\perp} \alpha)^n.$$

It is easy to show that $\Xi_A = 0$ if and only if $h_\beta = h_{\sigma_\alpha \beta}$. Similarly one may show (and here the condition that $\beta = \alpha + \sigma_{\beta^\perp} \alpha$ is used) that $\Xi_B = 0$ if and only if

$$(19) \quad (\alpha, \beta)h_\beta = (\alpha, \alpha - \beta)h_{\sigma_{\beta^\perp} \alpha}$$

The full sum is made up of sums of such paired-terms, and hence is zero. Repeating the argument for the terms that appear in equations (13) and (14) yields no further conditions.

□

Note that we have assumed that the h_α are Weyl invariant and hence for type A configurations the conditions in Proposition 16 are automatically satisfied with no extra conditions.

To illustrate this we begin with the simplest case, where $W = A_1$, which will reproduce the examples constructed earlier.

Example 21. $W = A_1$

- $|\mathfrak{U}| = 2$

Let $\mathfrak{U} = \mathcal{R}_{A_1} = \{\pm\alpha\}$ (normalized so that $(\alpha, \alpha) = 2$). Then conditions (17) and (18) imply that

$$h_\alpha = \frac{3}{8}, \quad h_{A_1}^\vee = \frac{3}{4}$$

(note their ratio is 2, which is the (dual) Coxeter number of A_1). The pole conditions are vacuous in this case. This gives solution (8).

- $|\mathfrak{U}| = 4$

Let $\mathfrak{U} = \{\pm\alpha, \pm\tilde{\alpha}\}$ with $(\alpha, \alpha) = 2, (\tilde{\alpha}, \tilde{\alpha}) = \nu$. Then conditions (17) and (18) imply that

$$\begin{aligned} 8h_\alpha + 2\nu^2 h_{\tilde{\alpha}} &= 3, \\ 2h_\alpha + \nu h_{\tilde{\alpha}} &= h_{\mathfrak{U}}^\vee. \end{aligned}$$

Without loss of generality let $h_\alpha = \frac{1}{2}$. Then

$$h_{\tilde{\alpha}} = -\frac{1}{2\nu^2}, \quad h_{\mathfrak{U}}^\vee = 1 - \frac{1}{2\nu}.$$

Again the pole conditions are vacuous. The choice $\nu = \frac{1}{2}$ is special ($h_{\mathfrak{U}}^\vee = 0$) and leads to the solution obtained in Corollary 10.

6.1. The case $\mathfrak{U} = \mathcal{R}_W$.

In this case it follows from general theory that (18) is satisfied for all Weyl groups (if $h_\alpha = 1$ for all roots then $h_{\mathfrak{U}}^\vee$ is just the dual Coxeter number of W). Since the quartic expression $\sum h_\alpha(\alpha, \mathbf{z})^4$ is Weyl invariant, by Chevellay's Theorem (Theorem 4) it may be written in terms of fundamental invariant polynomials of degree 2 and degree 4, i.e.

$$\sum h_\alpha(\alpha, \mathbf{z})^4 = A[s_2(\mathbf{z})]^2 + Bs_4(\mathbf{z})$$

if such polynomials exist. The quadratic polynomial s_2 exists for all groups W ; one may take $s_2(\mathbf{z}) = (\mathbf{z}, \mathbf{z})$. Invariant polynomials of degree 4 do not exist for $W = A_2, E_{6,7,8}, F_4, G_2$. Thus for these groups it follows immediately that (17) is satisfied. By direct calculation one may show that for the remaining Weyl groups, $W = A_{N \geq 3}, B_N, D_N$ (where such an invariant polynomial does exist) condition (17) fails, except for B_2 where it holds if a specific relationship between $h^{(long)}$ and $h^{(short)}$ exists. Thus, in general, for the three infinite families of groups, condition (17) fails and one has to append an extra set of Weyl-group invariant vectors in order to satisfy this condition: this will be done in the next section.

Since the constants h_α are Weyl invariant the analysis decomposes into cases labeled by the number of independent Weyl orbits:

- For $W = A_2, E_6, E_7, E_8$ one has a single Weyl orbit, so the constants h_α are all identical. The values of this constant, and the constant $h_{\mathfrak{U}}^\vee$ are tabulated below:

Weyl group	A_2	E_6	E_7	E_8
h_α	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{12}$
$h_{\mathfrak{U}}^\vee$	1	2	$\frac{9}{4}$	$\frac{5}{2}$

(note, $h_{\mathfrak{U}}^\vee/h_\alpha = (\text{dual})$ Coxeter number, as required).

- For $W = B_2, G_2, F_4$ one has two Weyl orbits, labeled by short and long roots. By direct computation one finds that conditions (17) and (18) are satisfied with the following data:

Weyl group	B_2	G_2	F_4
$h^{(long)}$	$\frac{1}{4}$	$\frac{1-h}{18}$	$\frac{3-h}{6}$
$h^{(short)}$	1	$\frac{3h-1}{6}$	$\frac{2h-3}{3}$
$h_{\mathfrak{U}}^\vee$	$\frac{3}{2}$	h	h

It remains now to check the conditions appearing in Proposition 16. It is well known that for a root system \mathcal{R}_W the configurations $\Pi_\alpha \cap \mathcal{R}_W$ are two dimensional root systems, namely one of $\mathcal{R}_{A_1 \times A_1}, \mathcal{R}_{A_2}, \mathcal{R}_{B_2}$ or \mathcal{R}_{G_2} , and all of these configurations are of type A. Hence by Lemma 20 these are elliptic \vee -systems and hence provide solutions of the WDVV equations.

6.2. The case $\mathfrak{U} = \mathcal{R}_W \cup \mathcal{R}_W^{irreg}$. We now turn our attention to the three infinite families, where one has to append an extra set of vectors to the standard roots in order to satisfy condition (18). Note the the Weyl groups A_1, A_2 and B_2 appear to be special in the sense that there are solutions with both $\mathfrak{U} = \mathcal{R}_W$ and $\mathfrak{U} = \mathcal{R}_W \cup \mathcal{R}_W^{irreg}$. For $A_{N \geq 3}$ and $B_{N \geq 3}$ condition (18) fails for $\mathfrak{U} = \mathcal{R}_W$.

6.2.1. The case $W = A_{N \geq 2}$.

Let $\mathbf{z} = \sum_{i=1}^{N+1} z^i e_i$, with $\sum_{i=1}^{N+1} z^i = 0$. With the later condition the following identities immediately follow:

$$\begin{aligned} \frac{1}{2} \sum_{i \neq j} (z^i - z^j)^2 - \frac{(N+1)}{2} \sum_i \{(z^i)^2 + (-z^i)^2\} &= 0, \\ \frac{1}{2} \sum_{i \neq j} (z^i - z^j)^4 - \frac{(N+1)}{2} \sum_i \{(z^i)^4 + (-z^i)^4\} &= 3 \left(\sum_i (z^i)^2 \right)^2. \end{aligned}$$

From these one may obtain the α and h_α satisfying conditions (17) and (18) on using the standard Euclidean inner product. Let

$$\begin{aligned} \alpha^{(ij)} &= e_i - e_j, \\ \beta^{(i)} &= \frac{1}{(N+1)} \left(N e_i - \sum_{j \neq i} e_j \right) \end{aligned}$$

(so $(\alpha^{(ij)}, \mathbf{z}) = z^i - z^j$ and $(\beta^{(i)}, \mathbf{z}) = z^i$). Note both these vectors lie on the hyperplane $\sum_{i=1}^{N+1} z^i = 0$. With these it follows that $\mathfrak{U} = \mathcal{R}_{A_N} \cup \mathcal{R}_{A_N}^{irreg}$ where:

$$\mathcal{R}_{A_N} = \{\alpha^{(ij)}, i \neq j\}, \quad h_\alpha = 1/2 \text{ if } \alpha \in \mathcal{R}_{A_N};$$

$$\mathcal{R}_{A_N}^{irreg} = \{\pm \beta^{(i)}, i = 1, \dots, N+1\}, \quad h_\alpha = -(N+1)/2 \text{ if } \alpha \in \mathcal{R}_{A_N}^{irreg}.$$

Note that \mathcal{R}_{A_N} is just the root system for A_N . The geometry of this configuration will now be discussed.

Let $\sigma_\alpha \beta$ denote the reflection of β in the plane perpendicular to α . Then

$$\begin{aligned} \sigma_{\alpha^{(ij)}} \beta^{(i)} &= \beta^{(i)} - \frac{2(\alpha^{(ij)}, \beta^{(i)})}{(\alpha^{(ij)}, \alpha^{(ij)})} \alpha^{(ij)}, \\ &= \beta^{(i)} - \alpha^{(ij)}, \\ &= \beta^{(j)}, \\ \sigma_{\alpha^{(ij)}} \beta^{(k)} &= \beta^{(k)}, \quad i, j, k \text{ distinct}. \end{aligned}$$

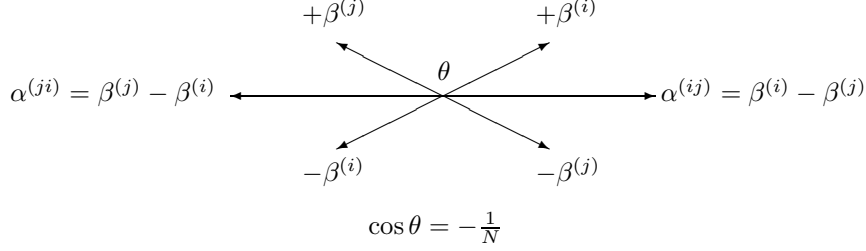
Thus the set $\mathcal{R}_{A_N}^{irreg}$ is invariant under the action of W (which is generated by reflections defined by the vectors in \mathcal{R}_{A_N}). Thus for $N \geq 3$ the Weyl orbit of an element of $\mathcal{R}_{A_N}^{irreg}$ is smaller (since $|\mathcal{R}_{A_N}^{irreg}| = 2(N+1)$) than the size of the orbit of a generic vector (which would be $|A_N| = (N+1)!$). Thus $\mathcal{R}_{A_N \geq 2}^{irreg}$ consists of the union of two irregular orbits

$$\mathcal{R}_{A_N \geq 2}^{irreg} \cong \{+\beta^{(i)} | i = 1, \dots, N+1\} \cup \{-\beta^{(i)} | i = 1, \dots, N+1\}.$$

There are certain degeneracies if $N = 1$ or 2 : if $N = 1$ then $\beta^{(1)} = -\beta^{(2)}$ and hence the set $\{\pm \beta^{(i)}\}$ double counts the vectors (this degeneracy was removed in the earlier discussion of the A_1 solution); if $N = 2$ then $|\mathcal{R}_{A_N}^{irreg}| = |\mathcal{R}_{A_N}| = (N+1)!$. In fact this case coincides with the G_2 example above, i.e. $\mathcal{R}_{G_2} \cong \mathcal{R}_{A_2} \cup \mathcal{R}_{A_2}^{irreg}$ for a specific value of the constant h , namely $h = 0$.

Note that since $(\beta^{(i)}, \alpha^{(jk)}) = 0$ and $(\beta^{(i)}, \alpha^{(ij)}) = 1$ the set $\mathcal{R}_{A_N}^{irreg}$ consists of vectors from the weight lattice of A_N . In terms of fundamental weights

$$\Delta_{(i)} = \sum_{r=1}^i e_r - \frac{i}{(N+1)} \sum_{r=1}^{N+1} e_r$$

FIGURE 1. The configuration $\mathfrak{U}_{A_N} \cap \text{Span}\{\beta^{(i)}, \beta^{(j)}\}$

one has

$$\mathcal{R}_{A_N}^{irreg} \cong \{\pm w(\Delta_{(N)}) : w \in W\}$$

(note that $\pm\Delta_{(1)}$ also lie in these two orbits). The orbits of other fundamental weights form other irregular orbits.

Furthermore, if $N \geq 3$ one obtains the configurations

$$\begin{aligned} \mathfrak{U} \cap \text{Span}\{\alpha^{(ij)}, \alpha^{(rs)}\} &= \mathcal{R}_{A_1 \times A_1}, \quad \{i, j\} \cap \{r, s\} \neq \emptyset, \\ \mathfrak{U} \cap \text{Span}\{\alpha^{(ij)}, \alpha^{(ik)}\} &= \mathcal{R}_{A_2}, \quad i, j, k \text{ distinct} \end{aligned}$$

together with the new configuration

$$\mathfrak{U} \cap \text{Span}\{\beta^{(i)}, \beta^{(j)}\} = \{\pm\beta^{(i)}, \pm\beta^{(j)}, \pm\alpha^{(ij)}\}.$$

The geometry of this new configuration is shown in Figure 1. This is precisely a type B configuration, and the condition (19) is satisfied, since $\alpha = \beta^{(i)}$, $h_\alpha = -(N+1)/2$ and $\beta = \alpha^{(ij)}$, $h_\beta = 1/2$. Hence by Lemma 20 we have an elliptic \vee -system and hence a solution to the WDVV equations.

6.2.2. The case $W = B_N$.

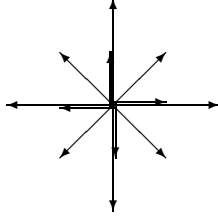
The dual prepotential for the Jacobi group orbit space $\Omega/J(B_N)$ may be calculated in the same way as the $\Omega/J(A_N)$ dual prepotential was derived in Theorem 6 (see also Example 27), and from this the set \mathfrak{U} and the constants h_α may be extracted.

Given this origin of the set one might expect that it should be related to the root system \mathcal{R}_{B_N} . It turns out that one may describe this set in two ways: either in terms of the root system \mathcal{R}_{BC_N} or in terms of the root system \mathcal{R}_{C_N} (which is, of course, dual to the root system \mathcal{R}_{B_N}) together with an irregular orbit $\mathcal{R}_{C_N}^{irreg}$.

Consider the following identities³:

$$\begin{aligned} \sum_{i \neq j} (z^i - z^j)^2 + (z^i + z^j)^2 + \sum_{i=1}^N (2z^i)^2 - 4N \sum_{i=1}^N (2z^i)^2 &= 0, \\ \sum_{i \neq j} (z^i - z^j)^4 + (z^i + z^j)^4 + \sum_{i=1}^N (2z^i)^4 - 4N \sum_{i=1}^N (2z^i)^4 &= 12 \left(\sum_i (z^i)^2 \right)^2. \end{aligned}$$

³Note the condition $\sum z_i = 0$ used in the last section is not used in this section.

FIGURE 2. The configuration $\mathfrak{U}_{B_N} \cap \text{Span}\{e_i, e_j\}$

On defining the inner product to be twice the standard Euclidean product (that is, $(\mathbf{z}, \mathbf{z}) = 2 \sum_i (z^i)^2$) one may obtain the α and h_α satisfying conditions (17) and (18).

In terms of the root system \mathcal{R}_{BC_N} , one has $\mathfrak{U}_{B_N} = \mathcal{R}_{BC_N}$ where

$$\mathcal{R}_{BC_N} = \left\{ \frac{1}{2}(\pm e_i \pm e_j), i \neq j \right\} \cup \{\pm e_i\} \cup \left\{ \pm \frac{1}{2} e_i \right\}$$

and

$$h_\alpha = \begin{cases} \frac{1}{2} & \text{if } \alpha \text{ is a long root,} \\ 1 & \text{if } \alpha \text{ is a middle root,} \\ -2N & \text{if } \alpha \text{ is a short root.} \end{cases}$$

Alternatively (and this provides a description that is closer to the A_N configuration above)

$$\begin{aligned} \mathfrak{U} &= \mathcal{R}_{C_N} \cup \mathcal{R}_{C_N}^{irreg}, \\ &= \mathcal{R}_{B_N}^\vee \cup \mathcal{R}_{B_N}^{\vee, irreg} \end{aligned}$$

where:

$$\mathcal{R}_{C_N} = \left\{ \frac{1}{2}(\pm e_i \pm e_j), i \neq j \right\} \cup \{\pm e_i\}, \quad h_\alpha = \begin{cases} 1 & \text{if } \alpha \text{ short} \\ 1/2 & \text{if } \alpha \text{ long} \end{cases} \text{ if } \alpha \in \mathcal{R}_{C_N};$$

$$\mathcal{R}_{C_N}^{irreg} = \{\pm \frac{1}{2} e_i\}, \quad h_\alpha = -2N \text{ if } \alpha \in \mathcal{R}_{C_N}^{irreg}.$$

As in the A_N case, $\mathcal{R}_{C_N}^{irreg}$ is an irregular orbit (a single orbit in this case):

$$\mathcal{R}_{C_N}^{irreg} = \{w(\Delta_{(N)}) | w \in W\}$$

for a certain fundamental weight $\Delta_{(N)}$.

In either case, the only new two dimensional configuration on vectors is $\mathfrak{U}_{C_N} \cap \text{Span}\{e_i, e_j\}$. This is shown in Figure 2, where the vectors of $\mathcal{R}_{C_N}^{irreg}$ have been displaced slightly for visual reasons (this is actually the BC_2 system). The proof that this is an elliptic \vee system follows the A_N case and will be omitted. It also follows from the Hurwitz space description that will be given in Section 8.2.

6.2.3. The case $W = D_N$.

The D_N configurations are combinatorially quite complicated, as, even at $N = 5$ several Weyl orbits of fundamental weights have to be appended to the basic root system \mathcal{R}_{D_N} in order to satisfy (17) and (18). Some of the resulting configurations $\Pi_\alpha \cap \mathcal{U}$ are not of Type A and Type B. This does not mean that the conditions in Proposition 16 must be false - there may be other reasons why the various terms could vanish.

In the $N = 4$ case $\mathcal{U} = \mathcal{R}_{D_4} \cup \mathcal{R}_{D_4}^{irreg}$ which coincides with the F_4 example considered above, with the long roots of F_4 being the roots of D_4 and the short roots being interpreted as the irregular orbits of the fundamental weights of D_4 . Clearly more work is required to construct an example of a D_N elliptic \vee -system.

The results in this section have been obtained on a case-by-case basis. It would be nice if there was a more abstract derivation of the results.

7. FROBENIUS-STICKELBERGER IDENTITIES

Hidden within the vanishing of the $\Delta^{(i)}$ are a number of interesting functional identities satisfied by the various third derivatives of the elliptic trilogarithm, the simplest of these reducing to 19th century ϑ -function identities. We build up to these by first considering the rational and trigonometric versions. Given non-zero $a, b, c \in \mathbb{C}$ such that $a + b + c = 0$ then

$$\frac{1}{a} \cdot \frac{1}{b} + \frac{1}{b} \cdot \frac{1}{c} + \frac{1}{c} \cdot \frac{1}{a} = 0$$

and

$$\cot(a) \cot(b) + \cot(b) \cot(c) + \cot(c) \cot(a) = 1.$$

Such identities are used in the direct verification that the rational (1) and trigonometric (2) prepotentials satisfy the WDVV equations. The elliptic version (where the dependence on τ has been suppressed for notational convenience) is

$$\left\{ \begin{array}{l} f^{(3,0)}(a)f^{(3,0)}(b) \\ + f^{(3,0)}(b)f^{(3,0)}(c) \\ + f^{(3,0)}(c)f^{(3,0)}(a) \end{array} \right\} - \left\{ f^{(2,1)}(a) + f^{(2,1)}(b) + f^{(2,1)}(c) \right\} = 0$$

Using (5) this may be written in terms of ϑ -functions⁴:

$$\frac{\vartheta'_1(a)}{\vartheta_1(a)} \frac{\vartheta'_1(b)}{\vartheta_1(b)} + \frac{\vartheta'_1(b)}{\vartheta_1(b)} \frac{\vartheta'_1(c)}{\vartheta_1(c)} + \frac{\vartheta'_1(c)}{\vartheta_1(c)} \frac{\vartheta'_1(a)}{\vartheta_1(a)} + \frac{1}{2} \left[\frac{\vartheta''_1(a)}{\vartheta_1(a)} + \frac{\vartheta''_1(b)}{\vartheta_1(b)} + \frac{\vartheta''_1(c)}{\vartheta_1(c)} \right] = \frac{1}{2} \frac{\vartheta'''_1(0)}{\vartheta'_1(0)}$$

where $a + b + c = 0$. With the identification $a = (\alpha, \mathbf{z})$, $b = (\beta, \mathbf{z})$, $c = -(\alpha + \beta, \mathbf{z})$ these identities may be seen as identities connected to the A_2 Coxeter group, with α and β being the positive roots. This immediately motivates the following:

⁴ This formula was found by the author during the researches that led to [27] and it has also appeared recently, with proof, in the work of Calaque, Enriques and Etingof [6]. However it is a classical formula; in terms of Weierstrass functions it is just the well known Frobenius-Stickelberger equation [34]

$$(\zeta(a) + \zeta(b) + \zeta(c))^2 = \wp(a) + \wp(b) + \wp(c), \quad (a + b + c = 0)$$

re-written in terms of ϑ -functions, an observation due to Prof. H.W.Braden.

Lemma 22. *Let \mathcal{R} be the root system for the 2-dimensional Coxeter groups A_2, B_2 or G_2 , with the standard normalization for α, β positive simple roots:*

$$\begin{aligned} A_2 : \quad & (\alpha, \alpha) = (\beta, \beta) = 2, (\alpha, \beta) = -1, \\ B_2 : \quad & (\alpha, \alpha) = 2, (\beta, \beta) = 1, (\alpha, \beta) = -1, \\ G_2 : \quad & (\alpha, \alpha) = 6, (\beta, \beta) = 2, (\alpha, \beta) = -3. \end{aligned}$$

Then

$$\sum_{\alpha \neq \beta \in \mathcal{R}^+} (\alpha, \beta) f^{(3,0)}(\mathbf{z}_\alpha, \tau) \cdot f^{(3,0)}(\mathbf{z}_\beta, \tau) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha f^{(2,1)}(\mathbf{z}_\alpha, \tau) = 0,$$

where:

- A_2 : $k_\alpha = 1$ for all roots;
- B_2 : $k_{short} = 2, k_{long} = 1$;
- G_2 : $k_{short} = 10, k_{long} = 6$.

The proof is entirely standard and is omitted. Many other functional identities may be derived using the same ideas. Rather than give a full list we present two of the A_2 identities:

$$\left\{ \begin{array}{l} f^{(3,0)}(x+y) [f^{(2,1)}(x) - f^{(2,1)}(y)] \\ + f^{(3,0)}(y) [f^{(2,1)}(x+y) - f^{(2,1)}(x)] \end{array} \right\} + f^{(1,2)}(x) - \frac{1}{2} f^{(1,2)}(y) + \frac{1}{2} f^{(1,2)}(x+y) = 0$$

and

$$\left\{ \begin{array}{l} f^{(3,0)}(x) [f^{(1,2)}(x+y) - f^{(1,2)}(y)] \\ + f^{(3,0)}(y) [f^{(1,2)}(x+y) - f^{(1,2)}(x)] \\ - \frac{2}{3} f^{(3,0)}(x+y) [f^{(1,2)}(x) + f^{(1,2)}(y)] \end{array} \right\} + \left\{ \begin{array}{l} \frac{2}{3} f^{(2,1)}(x+y) f^{(2,1)}(x) \\ + \frac{2}{3} f^{(2,1)}(x+y) f^{(2,1)}(y) \\ - \frac{8}{3} f^{(2,1)}(x) f^{(2,1)}(y) \end{array} \right\}$$

$$+ \frac{10}{9} f^{(0,3)}(x+y) = -\frac{1}{108} E_4(\tau).$$

Clearly there is much scope to investigate such neo-classical functional identities. More identities of these type may be found in [31].

8. JACOBI GROUP ORBIT SPACES

Mention has been made a number of times to Jacobi groups and their orbit spaces, but so far these have not been defined. In this section this is rectified and in addition the construction of the Frobenius manifold structure on such orbit spaces will be outlined. In particular, using an alternative description of such spaces as specific Hurwitz spaces we construct the dual prepotentials for the Weyl groups

A_N and B_N , thus proving that the examples of elliptic \vee -systems constructed earlier correspond to Jacobi group orbit spaces. This then motivates a conjecture for arbitrary Weyl group.

8.1. Jacobi groups and Jacobi forms. The material in this section will closely follow [2], which in turn relies heavily on the fundamental papers of Wirthmüller [33] and Eichler and Zagier [12]. We begin by the definition of a Jacobi form. These play the same role in the construction of the orbit space as the symmetric polynomials do in the original Saito construction - they provide coordinates on the orbit space.

Definition 23. *Let W be a finite Weyl group with root lattice Q and let \mathfrak{g} be the corresponding Lie algebra with Cartan subalgebra \mathfrak{h} . A Jacobi form of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}$ is a holomorphic function $\phi : \mathfrak{h} \oplus \mathbb{H} \rightarrow \mathbb{C}$ with the following properties:*

$$\begin{aligned} \phi(\mathbf{z} + \mathbf{q}, \tau) &= \phi(\mathbf{z}, \tau), \\ \phi(\mathbf{z} + \mathbf{q}\tau, \tau) &= e^{-2\pi i m(\mathbf{q}, \mathbf{z}) - \pi i m(\mathbf{q}, \mathbf{q})\tau} \cdot \phi(\mathbf{z}, \tau), \quad \text{for all } \mathbf{q} \in Q, \\ \phi\left(\frac{\mathbf{z}}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) &= (c\tau + d)^k \cdot e^{c\pi i m(\mathbf{z}, \mathbf{z})/(c\tau + d)} \cdot \phi(\mathbf{z}, \tau), \\ \phi(w.\mathbf{z}, \tau) &= \phi(\mathbf{z}, \tau), \quad \text{for all } w \in W \end{aligned}$$

and $\phi(\mathbf{z}, \tau)$ is a locally bounded function as $\Im m(\tau) \rightarrow +\infty$.

Such forms are the elliptic analogues of the W -invariant polynomials and they too satisfy a Chevalley-type theorem.

Following Bertola [2], one can define a new function $\phi(u, \mathbf{z}, \tau) = e^{mu}\phi(\mathbf{z}, \tau)$ defined on the Tits cone $\Omega \cong \mathbb{C} \oplus \mathfrak{h} \oplus \mathbb{H}$. This is also referred to as a Jacobi form and the space of Jacobi forms will be denoted $J_{k,m}^W$. The Jacobi group $J(\mathfrak{g})$ itself generates the above transformations. The full details are not required here: $J(\mathfrak{g})$ is the semi-direct product $\mathcal{W} \rtimes SL(2, \mathbb{Z})$ where $\mathcal{W} = W \rtimes H_{\mathcal{R}}$ where W is a Weyl group and $H_{\mathcal{R}}$ the Heisenberg group obtained from the root space \mathcal{R} of W . The precise definitions of the various actions may be found in [2, 33].

It is well known that the ring of modular forms is a free graded algebra over \mathbb{C} generated by the Eisenstein series E_4 and E_6 , i.e. $M_{\bullet} = \bigoplus_k M_k$, where the subspace of modular forms of weight k is

$$M_k = \mathbb{C}[E_4^a E_6^b, \forall a, b \in \mathbb{N} \text{ such that } 4a + 6b = k].$$

The ring of Jacobi forms is particularly nice; it satisfies an analogue of Chevalley's Theorem (Theorem 4):

Theorem 24. [33] *Given the Jacobi group associated to any finite dimensional simple Lie algebra \mathfrak{g} of rank N (except for possibly E_8):*

- *the bi-graded algebra of Jacobi forms $J_{\bullet, \bullet}^W = \bigoplus_{k,m} J_{k,m}^W$ is freely generated by $N + 1$ fundamental Jacobi forms $\{\phi_0, \dots, \phi_N\}$ over the graded ring of modular forms M_{\bullet} ,*

$$J_{\bullet, \bullet}^W = M_{\bullet}[\phi_0, \dots, \phi_N];$$

- each

$$\phi_j \in J_{-k(j), m(j)}^W$$

where $-k(j) \leq 0, m(j) > 0$ are defined as follows:

- $k(0) = 1$ and $k(j), j > 0$ are the degrees of the generators of the invariant polynomials in (4);
- $m(0) = 1$ and $m(j), j > 0$ are the coefficients in the expansion of the highest coroot $\tilde{\alpha}^\vee$,

$$\tilde{\alpha}^\vee = \sum_{j=1}^N m(j) \alpha_j^\vee$$

where $\tilde{\alpha}$ is the highest root and α_j^\vee a basis for \mathcal{R}^\vee .

Note that $J_{\bullet,0}^W \cong \mathbf{M}_\bullet$. It will also be useful to define $\phi_{-1} = \tau$, even though it is not a Jacobi form. These Jacobi forms become the coordinates on the orbit space $\Omega/J(\mathfrak{g})$. Before turning to the explicit construction of such forms we prove the following simple result on the Jacobian of the transformation between the two coordinate systems $\{u, \mathbf{z}, \tau\}$ and $\{\phi_{-1}, \phi_0, \dots, \phi_N\}$.

Proposition 25. *Let*

$$Jac(u, \mathbf{z}, \tau) = \frac{\partial\{\phi_{-1}, \phi_0, \dots, \phi_N\}}{\partial\{u, \mathbf{z}, \tau\}}.$$

Then Jac has the following transformation properties:

$$\begin{aligned} \frac{1}{2\pi i} \frac{\partial}{\partial u} Jac(u, \mathbf{z}, \tau) &= h^\vee Jac(u, \mathbf{z}, \tau), \\ Jac(u, \mathbf{z} + \mathbf{q}, \tau) &= Jac(u, \mathbf{z}, \tau), \\ Jac(u, \mathbf{z} + \mathbf{q}\tau, \tau) &= e^{-2\pi i h^\vee(\mathbf{q}, \mathbf{z}) - \pi i h^\vee(\mathbf{q}, \mathbf{q})} \cdot Jac(u, \mathbf{z}, \tau), \\ Jac(u, \mathbf{z}, \tau + 1) &= Jac(u, \mathbf{z}, \tau), \\ Jac\left(u, \frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right) &= \tau^{-|\mathcal{R}_W^+|} e^{\pi i h^\vee(\mathbf{z}, \mathbf{z})/\tau} \cdot Jac(u, \mathbf{z}, \tau), \\ Jac(u, w, \mathbf{z}, \tau) &= \det(w) \cdot Jac(u, \mathbf{z}, \tau), \end{aligned}$$

where h^\vee is the dual Coxeter number and $|\mathcal{R}_W^+|$ the number of positive roots. Moreover, up to an overall constant,

$$(20) \quad Jac(u, \mathbf{z}, \tau) = e^{2\pi i h^\vee u} \prod_{\alpha \in \mathcal{R}_W^+} \frac{\vartheta_1(\mathbf{z}_\alpha, \tau)}{\vartheta_1'(0, \tau)}.$$

These transformation properties may be elevated to a definition of an anti-invariant Jacobi form. This result is the elliptic version of the well known result $Jac(\mathbf{z}) = \prod_{\alpha \in \mathcal{R}_W^+} \mathbf{z}_\alpha$ for Coxeter groups.

Proof By definition, the Jacobian is a determinant, so by using properties of the determinant, together with the transformation properties of the individual Jacobi forms given in Proposition 23 the result follows. Various Lie-theory results are used, such as

$$h^\vee = \sum_{i=0}^N m(i), \quad |\mathcal{R}_W^+| = \sum_{i=1}^N (k(i) - 1),$$

proofs of which may be found in Kac [17].

To prove (20) one first proves that the right-hand-side has the same transformation properties as Jac . Therefore their ratio transformations like a $J_{0,0}^W$ -Jacobi form (the analytic properties following from those of the ϑ_1 -function, such as its entire property). But $J_{0,0}^W \cong M_0$ and there are no non-trivial degree 0 modular forms and hence the ratio must be a constant. \square

Further properties of the forms may be found in [2, 33]. For the A_N and B_N cases there is a very compact way to study the forms by combining them into a generating function. The invariant polynomials for the A_N -Coxeter group may be obtained via a generating function (a result due to Viète)

$$\prod_{i=0}^N (v - z_i) \Big|_{\sum z^i = 0} = v^{N+1} + \sum_{r=0}^{N-1} (-1)^{N+1-r} s_{r+1}(\mathbf{z}) v^r.$$

Similarly, the A_N Jacobi forms may be obtained [2] from a similar expansion of

$$(21) \quad \lambda^{A_N}(v) = e^{2\pi i u} \frac{\prod_{i=0}^N \vartheta_1(v - z_i, \tau)}{\vartheta_1(v, \tau)^{N+1}} \Big|_{\sum z^i = 0}$$

as a sum of Weierstrass \wp functions and their derivatives, their coefficients being the A_N -Jacobi forms. Using the embedding $B_N \subset A_{2N-1}$ one may obtain a generating function for the B_N -Jacobi forms:

$$(22) \quad \lambda^{B_N}(v) = e^{2\pi i u} \frac{\prod_{i=1}^N \vartheta_1(v - z_i, \tau) \vartheta_1(v + z_i, \tau)}{\vartheta_1(v, \tau)^{2N}}$$

These generating functions are not just formal objects, they are holomorphic maps from the complex torus to the Riemann sphere. This means one can use a Hurwitz space construction to calculate the dual prepotential.

8.2. Hurwitz spaces. Let $H_{g,N}(k_1, \dots, k_l)$ be the Hurwitz space⁵ of equivalence classes $[\lambda : \mathcal{L} \rightarrow \mathbb{P}^1]$ of N -fold branched coverings $\lambda : \mathcal{L} \rightarrow \mathbb{P}^1$, where \mathcal{L} is a compact Riemann surface of genus g and the holomorphic map λ of degree N is subject to the following conditions:

- it has M simple ramification points $P_1, \dots, P_M \in \mathcal{L}$ with distinct *finite* images $l_1, \dots, l_M \in \mathbb{C} \subset \mathbb{P}^1$;
- the preimage $\lambda^{-1}(\infty)$ consists of l points: $\lambda^{-1}(\infty) = \{\infty_1, \dots, \infty_l\}$, and the ramification index of the map λ at the point ∞_j is k_j ($1 \leq k_j \leq N$).

(We define the ramification index at a point as the number of sheets of the covering which are glued together at this point. A point ∞_j is a ramification point if and only if $k_j > 1$. A ramification point is simple if the corresponding ramification index equals 2.) The Riemann-Hurwitz formula implies that the dimension of this space is $M = 2g + l + N - 2$. One has also the equality $k_1 + \dots + k_l = N$. Two branched coverings $\lambda_1 : \mathcal{L}_1 \rightarrow \mathbb{P}^1$ and $\lambda_2 : \mathcal{L}_2 \rightarrow \mathbb{P}^1$ are said to be equivalent if there exists a biholomorphic map $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ such that $\lambda_2 f = \lambda_1$.

We also introduce the covering $\hat{H}_{g,N}(k_1, \dots, k_l)$ of the space $H_{g,N}(k_1, \dots, k_l)$ consisting of pairs

$$< [\lambda : \mathcal{L} \rightarrow \mathbb{P}^1] \in H_{g,N}(k_1, \dots, k_l), \{a_\alpha, b_\alpha\}_{\alpha=1}^g >,$$

⁵ Dubrovin [9] uses a slightly different notation. In his notation the Hurwitz space is $H_{g;k_1-1, \dots, k_l-1}$.

where $\{a_\alpha, b_\alpha\}_{\alpha=1}^g$ is a canonical basis of cycles on the Riemann surface \mathcal{L} . The spaces $\hat{H}_{g,N}(k_1, \dots, k_l)$ and $H_{g,N}(k_1, \dots, k_l)$ are connected complex manifolds and the local coordinates on these manifolds are given by the finite critical values of the map λ . For $g = 0$ the spaces $\hat{H}_{g,N}(k_1, \dots, k_l)$ and $H_{g,N}(k_1, \dots, k_l)$ coincide.

The various metric and multiplication tensors are given in terms of this holomorphic map $\lambda : \mathcal{L} \rightarrow \mathbb{P}^1$ (also known as the superpotential) by the following:

Theorem 26. *The intersection form and dual multiplication on the Hurwitz space $H_{g,N}(k_1, \dots, k_l)$ are given by the following residue formulae:*

$$\begin{aligned} g(\partial', \partial'') &= \sum_{d\lambda=0} \operatorname{res} \frac{\partial'(\log \lambda(v)dv) \partial''(\log \lambda(v)dv)}{d \log \lambda(v)}, \\ c^*(\partial', \partial'', \partial''') &= \frac{1}{2\pi i} \sum_{d\lambda=0} \operatorname{res} \frac{\partial'(\log \lambda(v)dv) \partial''(\log \lambda(v)dv) \partial'''(\log \lambda(v)dv)}{d \log \lambda(v)}. \end{aligned}$$

Here ∂, ∂' and ∂'' are arbitrary vector fields on the Hurwitz space $H_{g,N}(k_1, \dots, k_l)$.

The formula for g appeared in [9] while the formula for c^* follows immediately from the results in [10]. Note that with the specific dependence of u in the superpotentials (21) and (22) we have normalized g and c^* so that ∂_u is the unity vector field (rather than $\frac{1}{2\pi i} \partial_u$). Thus $c^*(\partial_u, \partial', \partial'') = g(\partial', \partial'')$. This also avoids a proliferation of $(2\pi i)$ -factors in the final result.

Certain Hurwitz spaces are isomorphic to certain orbit spaces [9]. For example,

$$\begin{aligned} \mathbb{C}^N / A_N &\cong H_{0,N+1}(N+1), \\ \mathbb{C}^{N+1} / \tilde{A}_N^{(k)} &\cong H_{0,N+1}(k, N-k), \\ \Omega / J(A_N) &\cong H_{1,N+1}(N+1). \end{aligned}$$

Thus the tower of generalizations mentioned in the introduction has a unified description, at least for the A_N -cases, in terms of the theory of Hurwitz spaces. This also leads to a way to expand the tower further via higher genus Hurwitz spaces, the most natural being the space $H_{g,N+1}(N+1)$. The B_N examples come from introducing a \mathbb{Z}_2 grading onto the Hurwitz space (e.g. the superpotentials above have a $z \leftrightarrow -z$ symmetry).

Example 27.

- (a) *Using the superpotential (21) one obtains the intersection form and (dual) prepotential for the A_N -Jacobi group orbit space in Theorem 6 above [27]:*

$$\begin{aligned} g &= 2du d\tau - \sum_{i=0}^N (dz^i)^2 \Big|_{\sum_{j=0}^N z^j = 0} \\ F^*(u, \mathbf{z}, \tau) &= \frac{1}{2} \tau u^2 - \frac{1}{2} u \sum_{i=0}^N (z^i)^2 \Big|_{\sum_{j=0}^N z^j = 0} \\ &\quad + \frac{1}{2} \sum_{i \neq j} f(z^i - z^j, \tau) - (N+1) \sum_i f(z^i, \tau) \end{aligned}$$

where this function is evaluated on the plane $\sum_{i=0}^N z^i = 0$.

- (b) Using the superpotential (22) one obtains the intersection form and (dual) prepotential for the B_N -Jacobi group orbit space:

$$\begin{aligned}
g &= 2du d\tau - 2 \sum_{i=1}^N (dz^i)^2, \\
F^*(u, \mathbf{z}, \tau) &= \frac{1}{2} \tau u^2 - 2u \sum_{i=0}^N (z^i)^2 \\
&\quad + \sum_{i \neq j} \{f(z^i + z^j, \tau) + f(z^i - z^j, \tau)\} \\
&\quad + \sum_i f(2z^i, \tau) - 2N \sum_i f(z_i, \tau).
\end{aligned}$$

Combining this with the earlier results on elliptic \vee -systems gives:

Theorem 28. *The elliptic \vee -systems given in sections (6.2.1) and (6.2.2) define prepotentials that are the almost-dual prepotentials associated to the A_N and B_N Jacobi group orbit spaces.*

The form of this result, coupled with the examples of elliptic \vee -systems leads to the following conjecture:

Conjecture 29. *Let W be a Weyl group. For the Jacobi group orbit space $\Omega/J(W)$ the dual prepotential takes the form (15) with $h_{\mathfrak{U}}^\vee = 0$. Furthermore, $\mathfrak{U} = \mathcal{R}_W \cup \mathcal{R}_W^{irreg}$ (or its dual) and where*

$$\mathcal{R}_W^{irreg} = \{w(\Delta) | w \in W\} \quad \text{or} \quad \mathcal{R}_W^{irreg} = \{\pm w(\Delta) | w \in W\}$$

for some weight vector Δ .

The conjecture seems plausible. It is true for the A_N examples and the B_N examples (if one uses the dual root system) and if all orbit spaces are to behave in the same generic way in the trigonometric limit then one must have $h_{\mathfrak{U}}^\vee = 0$ from the results of section 5.1.

One possible approach to proving this conjecture would be to show that if a prepotential F lies at the fixed point of the involutive symmetry then so does the corresponding almost dual prepotential. Since this is true for the Jacobi group examples this would then prove the first part of the conjecture, but not the second part on the structure of the set \mathfrak{U} . The Saito construction of Jacobi groups has recently been studied in detail [29]. Perhaps a formulation of a dual version of the result would provide a proof of the conjecture.

Within the class of elliptic \vee -systems there remains the problem of constructing examples with $h_{\mathfrak{U}}^\vee = 0$. For $W = G_2, F_4$ one may set $h = 0$, but for $E_{6,7,8}$ one would have to append an \mathcal{R}_W^{irreg} set of vectors. These cases also remain problematical. If $h = 0$ in the G_2 case one obtains a dual prepotential that is actually the dual prepotential for the A_2 Jacobi group orbit space, leaving a problem as to what the correct G_2 solution would be. This case lies in the so-called co-dimension one case and deserves closer study. The G_2 Jacobi forms have also been constructed explicitly [2] so it may be possible to find the dual prepotential in this case by direct calculation. It is also possible the the dual prepotential is the same in these two cases: the reconstruction of the Frobenius manifold from the dual picture requires additional data besides the almost dual prepotential.

9. COMMENTS

The idea of an elliptic \vee -system may clearly be studied further. As well as the obvious question on the relationship between the functional ansatz and Jacobi group orbit spaces summarized in Conjecture 29, there are many other questions and problems that could be addressed. Given a complex Euclidean \vee -system one may study their restriction to lower dimensions and the conditions required for the restricted system to also be a complex Euclidean \vee -system. Clearly the same question can be asked for elliptic \vee -systems. Examples along this line may be obtained from the restriction of the A_N and B_N Jacobi-group spaces to discriminants. This is achieved by introducing multiplicities into the A_N superpotential (21),

$$\lambda(p) = e^{2\pi i u} \prod_{i=0}^m \left(\frac{\vartheta_1(v - z_i, \tau)}{\vartheta_1(v, \tau)} \right)^{k_i},$$

where $\sum_{i=0}^m k_i = N + 1$, $\sum_{i=0}^m k_i z_i = 0$, or on more general Hurwitz spaces $H_{1,N}(n_1, \dots, n_m)$ and their discriminants. Partial results have been obtained in [26], and these provide further examples of elliptic \vee -systems. In fact, interesting examples of \vee -system can be found by looking on the induced structures on discriminants [14, 30] and clearly the same ideas could be applied here.

Possible applications of these solutions should come from Seiberg-Witten theory and the perturbative limits of such theories. This link is well known for rational and trigonometric solutions, and the interpretation of the elliptic solutions found in [27] in terms of a 6-dimensional field theory has been given in [3], and one would expect similar results for the more general solutions constructed here (though [3] does use the existence of a superpotential which is lacking for general solutions constructed here).

The tower of generalizations mentioned in the introduction clearly does not have to stop at elliptic solutions. An arbitrary Hurwitz space $H_{G,N}(k_1, \dots, k_l)$ carries the structure of a Frobenius manifold and hence an almost-dual structure. An interesting question is whether or not there is an orbit space construction for these more general spaces:

$$\begin{array}{ccccccc} H_{0,N}(N) & \longrightarrow & H_{0,N}(k, N-k) & \longrightarrow & H_{1,N}(N) & \longrightarrow & \dots \longrightarrow H_{g,N}(N) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}^N/A_N & \longrightarrow & \mathbb{C}^{N+1}/\tilde{A}_N^{(k)} & \longrightarrow & \Omega/J(A_N) & \longrightarrow & \dots \longrightarrow \text{orbit space structure?} \end{array}$$

It seems sensible to conjecture that such an orbit space exists. One would expect Siegel modular forms to play a role instead of the modular forms used here. Higher genus Jacobi forms certain have been studied, but their use has yet to percolate into the theory of integrable systems. The development, and applications of, the neo-classical ϑ -function identities studied in Section 7 remains to be done systematically. Certain higher genus analogues of these identities certainly exist, since there exist almost-dual prepotentials on these Hurwitz spaces which, by construction, satisfy the WDVV equations. In the genus 0 and genus 1 cases, the prepotential is very closely related to the prime form on the Riemann surface. This may be the starting point for the development of a functional ansatz for the higher genus cases. Central

to the results presented here are the quasi-periodicity and modularity properties of the elliptic polylogarithm, and these were obtained from the analytic properties of this function; the only role the analytic properties play were in the development of these transformation properties. It would be attractive if one could obtain these directly from the geometric properties of the prime form. This approach could then be used in the higher genus case where the analytic properties are likely to be considerably more complicated.

Mention has been made already of the beautiful paper [6]. It would be interesting to see if the ideas developed here could be used in the study of KZ and Dunkl-type systems. The idea would be to study objects such as

$$\sum_{\alpha, \beta \in \mathfrak{U}} \left[f^{(3,0)}(\mathbf{z}_\alpha) s_\alpha, f^{(3,0)}(\mathbf{z}_\beta) s_\beta \right]$$

where s_α and s_β are shift operators. Conjecturally this quadratic term would be related to linear terms in the function $f^{(2,1)}$. The rational limit would then coincide with the classical work of Dunkl [5]. Such a development would be different to the elliptic Dunkl operators in the pioneering work of Buchstaber et al. [4]. For a preliminary discussion of these ideas, see [31].

Finally, one thing that has been learnt from this work is that on going from rational and trigonometric structure related to a Weyl group W via the root system \mathcal{R}_W to elliptic structures, generalizations based entirely on the use of the root system \mathcal{R}_W alone may not suffice.

ACKNOWLEDGMENTS

I would like to thank Harry Braden, Misha Feigin and Andrew Riley for their comments on this paper. I am also very grateful to the referee for pointing out certain errors in the original version, and for his/her careful reading of the manuscript - this has resulted in much improved final version.

REFERENCES

- [1] Beilinson, A. and Levin, A., *The Elliptic Polylogarithm*, in *Motives* (ed. Jannsen, U., Kleiman, S., Serre, J.-P.), Proc. Symp. Pure Math. vol **55**, Amer. Math. Soc., (1994), Part 2, 123-190.
- [2] Bertola, M. *Frobenius manifold structure on orbit space of Jacobi groups; Parts I and II*, Diff. Geom. Appl. **13**, (2000), 19-41 and **13**, (2000), 213-23.
- [3] Braden, H.W., Marshakov, A., Mironov, A. and Morozov, A., *WDVV equations for 6d Seiberg-Witten theory and bi-elliptic curves*, Acta Applicandae Mathematicae **99** (3) (2007) 223-244.
- [4] Buchstaber, V.M., Felder, G. and Veselov, A.P., *Elliptic Dunkl operators, root systems, and functional equations* Duke Math. J. **76:3** (1994), 885-911.
- [5] Dunkl, C. *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989) 167-183.
- [6] Calaque, D., Enriquez, B. and Etingof, P., *Universal KZB equations I: The Elliptic Case* arXiv:math.QA/0702670.
- [7] Chalykh, O. and Veselov, A.P., *Locus configurations and V-systems*, Phys. Lett. **A285** 339-349, 2001.
- [8] Cherednik, I., *Double Affine Hecke Algebras*, LMS Lecture Note Series vol. 319, Cambridge University Press (2005).
- [9] Dubrovin, B., *Geometry of 2D topological field theories in Integrable Systems and Quantum Groups*, ed. Francaviglia, M. and Greco, S., Springer lecture notes in mathematics, **1620**, 120-348.

- [10] Dubrovin, B., *On almost duality for Frobenius manifolds* in *Geometry, topology, and mathematical physics*, 75–132, Amer. Math. Soc. Transl. Ser. 2, 212, Amer. Math. Soc., Providence, RI, 2004.
- [11] Dubrovin, B. and Zhang, Y., *Extended affine Weyl groups and Frobenius manifolds*, *Compositio Math.* **111** (1998), 167–219.
- [12] Eichler, M. and Zagier, D., *The Theory of Jacobi Forms*, Birkhäuser, 1985 (*Progress in Mathematics* Vol. 55).
- [13] Feigin, M.V., *Trigonometric solutions of WDVV equations and generalized Calogero-Moser-Sutherland systems*, *SIGMA* **5** (2009), 088, 10 pages.
- [14] Feigin, M.V. and Veselov, A.P., *Logarithmic Frobenius structures and Coxeter discriminants*, *Advances in Mathematics* **212:1** (2007) 143–162.
- [15] Feigin, M.V. and Veselov, A.P., *On the geometry of V-systems*, in *Geometry, Topology, and Mathematical Physics*, Amer. Math. Soc. Transl. Ser. 2, Vol. 224, Amer. Math. Soc., Providence, RI, 2008, 111–123.
- [16] Humphreys, J.E., *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics vol 9, Springer-Verlag (1992), *Reflection Groups and Coxeter Groups*, Cambridge studies in advanced mathematics vol 29, Cambridge University Press (1990).
- [17] Kac, V.G., *Infinite dimensional Lie algebras*, Cambridge University Press (1990).
- [18] Kawai, T., *String duality and enumeration of curves by Jacobi forms in Integrable systems and algebraic geometry* (Kobe/Kyoto, 1997), 282–314, World Sci. Publishing, River Edge, NJ, 1998.
- [19] Levin, A., *Elliptic polylogarithms: an analytic theory* *Compositio Math.* 106 (1997), no. 3, 267–282.
- [20] Marshakov, A., Mironov, A., and Morozov, A., *A WDVV-like equation in $N = 2$ SUSY Yang-Mills theory*, *Phys. Lett. B* **389** (1996) 43–52.
- [21] Marshakov, A. and Mironov, A., *5d and 6d Supersymmetric Gauge Theories: Prepotentials from Integrable Systems*, *Nucl.Phys. B* 518 (1998) 59–91.
- [22] Martini, R. and Gragert, P.K.H., *Solutions of WDVV equations in Seiberg-Witten theory from root systems*, *J. Nonlinear. Math. Phys.* **61**(1) (1999) 1–4.
- [23] Martini, R., and Hoevenaars, L.K., *Trigonometric Solutions of the WDVV Equations from Root Systems*, *Lett.Math.Phys.* **65** (2003) 15–18.
- [24] Olshanetsky, M.A. and Perelomov, A.M., *Quantum integrable systems related to Lie algebras*, *Phys. Rep.* **98** (1989), 1–18.
- [25] Ramakrishnan, D., *On the monodromy of higher logarithms*, *Proc. Amer. Math. Soc.* 85 (1982), no. 4, 596–599.
- [26] Riley, A., *Frobenius manifolds: caustic submanifolds and discriminant almost duality*, Ph.D. thesis, Hull University, 2007.
- [27] Riley, A. and Strachan, I.A.B., *Duality for Jacobi Group Orbit Spaces and Elliptic Solutions of the WDVV Equations*, *Lett. Math. Phys.* **77:3** (2006) 221–234.
- [28] Saito, K., *On a linear structure of a quotient variety by a finite reflection group*, Preprint RIMS-288 (1979).
- [29] Satake, I., *Frobenius manifolds for elliptic root systems*, arXiv:math/0611553
- [30] Strachan, I.A.B., *Frobenius manifolds: natural submanifolds and induced bi-Hamiltonian structures*, *Differential Geometry and its Applications*, 20 (2004), 67–99.
- [31] Strachan, I.A.B., *Differential and functional identities for the elliptic trilogarithm*, *SIGMA* **5**, (2009), 031, 12 pages.
- [32] Veselov, A.P., *Deformations of the root systems and new solutions to generalized WDVV equations*, *Phys. Lett. A* **261** (1999) 297–302.
- [33] Wirthmüller, K., *Root systems and Jacobi forms*, *Compositio Mathematica* **82** (1992) 293–354.
- [34] Whittaker, E.T. and Watson, G.N., *A course of Modern Analysis*, Cambridge University Press (any edition).
- [35] Zagier, D. *The Bloch-Wigner-Ramakrishnan polylogarithm function*, *Math. Ann.* **286** (1990) 613–624.